

Chaplygin's Sphere

J.J. Duistermaat*

Abstract

Chaplygin [9] proved the integrability by quadratures of a round sphere, rolling without slipping on a horizontal plane, with center of mass at the center of the sphere, but with arbitrary moments of inertia. Although the system is integrable in every sense of the word, it neither arises as a Hamiltonian system, nor is the integrability an immediate consequence of the symmetries. On the other hand, the constants of motion are obtained as a consequence of Noether's principle, cf. Section 1 and 2. The system also turns out to be related to a Hamiltonian system, the geodesic flow on the Euclidean motion group for a left invariant metric, cf. Subsection 9.3.

In this paper we analyse the global dynamics of Chaplygin's sphere. In the process we will explain almost all of Chaplygin's results. Readers who are mainly interested in these may consult Sections 1, 2, 3, 7, Subsections 9.2, 11.2, 11.3, 11.5, and Section 12. These can be read independently from the rest of the paper with some exceptions, such as Subsection 9.2 in which Subsection 9.1 has been used. At the end of each section we describe in a subsection "Chaplygin" the relation between our text and Chaplygin's.

We also obtain some new results, such as the proof in Section 4 that the level sets of the constants of motion in the reduced phase space are two-dimensional tori. In Section 8 we prove that, after a suitable time reparametrization, the rotational motion is quasi-periodic on two-dimensional tori. After suitable completion of the level surfaces, this is also true for the complexified system, cf. Section 10. This shows that the rotational motion is algebraically integrable according to the definition of Adler and van Moerbeke. In Subsection 11.4 it is explained how this also follows, in a quite different way, from Chaplygin's integration in terms of hyperelliptic integrals.

1 Noether's Principle for Nonholomic Systems

We use the equations of motion for a system with nonholonomic constraints as given by d'Alembert's principle

$$\langle [L]^\gamma(t), v \rangle = 0 \text{ for every } v \in C_{\gamma(t)}, \quad (1.1)$$

*A large part of this work has been done during a sabbatical leave in Berkeley, in the fall of 1994, partially supported by AFOSR Contract AFO F 49629-92. A more recent version was prepared in July 2000, as a chapter in the planned book "The geometry of Nonholonomically Constrained Systems", together with R.H. Cushman and J. Śniaticky.

as described in [4, Ch. 1, Sec. 2.5]. We may assume that $Q_0 = Q$. Let $s \mapsto \gamma_s$ be a smooth family of smooth curves in Q , with $\gamma = \gamma_0$, $\delta(t) := \partial \gamma_s(t) / \partial s|_{s=0}$, for which we this time do not assume that $\delta(a) = 0$ and $\delta(b) = 0$. Write

$$j^\delta(t) := \sum_i \frac{\partial L(\gamma(t), v)}{\partial v^i} \Big|_{v=\gamma'(t)} \delta^i(t)$$

for the δ -component of the momentum, which is a coordinate-invariant quantity. The classical variational equation, which is obtained by a partial integration, reads

$$\frac{d}{ds} \int_a^b L(\gamma_s(t), \gamma'_s(t)) dt \Big|_{s=0} = - \int_a^b \langle [L]^\gamma(t), \delta(t) \rangle dt + j^\delta(b) - j^\delta(a). \quad (1.2)$$

If one differentiates (1.2) with respect to b , then one obtains the equivalent form

$$\frac{d}{dt} j^\delta(t) = \frac{d}{ds} L(\gamma_s(t), \gamma'_s(t)) \Big|_{s=0} + \langle [L]^\gamma(t), \delta(t) \rangle. \quad (1.3)$$

The second term in the right hand side of (1.3) can be viewed as the δ -component of the reaction force acting on the system. It is equal to zero if γ satisfies d'Alembert's principle (1.1) and $\delta(t) \in C_{\gamma(t)}$, which means in words that $\delta(t)$ is a virtual displacement.

The form (1.3) of the variational equations is due to Emmy Noether [33], in a version where the independent variable t is replaced by a finite number of real variables and L is a smooth function on a jet bundle of arbitrary order. In the case that the first term in the right hand side of (1.3) is equal to zero, the formula (1.3), is the “momentum equation” of Bloch e.a. [6, Thm. 4.5].

Suppose that w is a smooth vector field on Q such that $\frac{d}{ds} \gamma_s(t)|_{s=0} = w(\gamma(t))$. Let \widehat{w} denote the lift of w to TQ , the vector field \widehat{w} on TQ such that

$$e^s \widehat{w} = T(e^s w), \quad (1.4)$$

if $e^s w$ denotes the flow after time s of the vector field w . In local coordinates \widehat{w} is given by

$$\widehat{w}(x, v) = (w(x), Dw(x) \cdot v), \quad (x, v) \in TQ, \quad (1.5)$$

where $Dw(x)$ denotes the matrix $\partial w^i(x) / \partial x^j$. With this notation, the first term in the right hand side of (1.3) is equal to the derivative of L at $(\gamma(t), \gamma'(t))$ in the direction of \widehat{w} . This leads to the following version of Noether's principle for variational systems with nonholonomic constraints.

Lemma 1.1 *Let L be a smooth function on TQ , of which C is a smooth vector subbundle. Let w be a smooth vector field on Q with the following properties*

- i) *w is a virtual displacement, which means that w is a section of C .*
- ii) *At each point of C , the derivative of L in the direction of \widehat{w} is equal to zero, where \widehat{w} is the lift of w to TQ as defined by (1.4).*

Then the w -component of the momentum is constant along every solution of (1.1).

If there are no constraints, when $C = TQ$, then condition i) is void and ii) is equivalent to the condition that L is invariant under the flow of the vector field \widehat{w} in TQ , which is equal to the tangent lift of the flow of w in Q . In this case Lemma 1.1 is due to Emmy Noether [33].

Question 1.2 Can all the constants of motion in Chaplygin [8] be obtained as applications of Lemma 1.1? ⊗

Remark 1.3 Lemma 1.4 below leads to the warning that in the non-integrable case the condition ii), under the assumption that i) holds, is not a property of only the restriction of L to C , because at the points of C the vector field \widehat{w} need not be tangent to C . ⊗

Lemma 1.4 *For each section $w : Q \rightarrow C$ of C the vector field \widehat{w} is tangent to C if and only if the subbundle C of TQ is integrable.*

Proof Let ψ^s be the flow of w . Then the condition that the lift of w is tangent to C is equivalent to the condition that the mappings $T\psi^s$ leave C invariant, or that these mappings send sections of C to sections of C . This in turn is equivalent to the condition that $[w, u]$ is a section of C for every section u of C . That this holds for every section w of C is one of the equivalent Frobenius conditions for the integrability of C . □

1.1 Chaplygin

The version of Lemma 1.1 with nonholonomic constraints can be found in Arnol'd [4, p. 82], with condition ii) replaced by the somewhat stronger condition that L is \widehat{w} -invariant. Two applications have been given in Arnol'd [4, p. 83, 84], the first with a reference to Chaplygin [8] and the second with a reference to Chaplygin [9]. In Chaplygin [9] the constants of motion have been described as an application of [8].

2 Noether's Principle for Chaplygin's Sphere

The position of a rigid body is given by a pair (A, a) , with $A \in \text{SO}(3)$ and $a \in \mathbf{R}^3$. Thus, if $x \in \mathbf{R}^3$ is the position of a material point of the body in its reference position, then $y = Ax + a$ is the position of the corresponding point in the moving body. If s is the position on the surface S of the body in the reference position, such that $p = As + a$ is the point of contact of the moving body with the surface P on which the body is rolling, then the condition of rolling without slipping means that

$$\dot{A}s + \dot{a} = 0, \tag{2.1}$$

meaning that the at the point of contact the corresponding material point of the body is at rest. Correspondingly, (\tilde{A}, \tilde{a}) is a virtual displacement if and only if

$$\tilde{A}s + \tilde{a} = 0. \quad (2.2)$$

If μ denotes the mass distribution of the body in the reference position, which is a finite Borel measure on \mathbf{R}^3 , then the kinetic energy of the moving body is given by

$$T = \int_{\mathbf{R}^3} \frac{1}{2} \langle \dot{A}x + \dot{a}, \dot{A}x + \dot{a} \rangle \mu(dx). \quad (2.3)$$

It follows that the (\tilde{A}, \tilde{a}) -component of the momentum is equal to

$$j^{(\tilde{A}, \tilde{a})} = \int_{\mathbf{R}^3} \langle \dot{A}x + \dot{a}, \tilde{A}x + \tilde{a} \rangle \mu(dx). \quad (2.4)$$

Let $\nu \in \mathbf{R}^3$ be the unique vector such that

$$\tilde{A}z = \nu \times (Az), \quad z \in \mathbf{R}^3 \quad (2.5)$$

— note that this corresponds to the *right* trivialization of the tangent bundle. If the condition (2.2) holds, meaning that (\tilde{A}, \tilde{a}) is a virtual displacement, then

$$j^{(\tilde{A}, \tilde{a})} = \int_{\mathbf{R}^3} \langle \dot{y}, \tilde{A}(x - s) \rangle \mu(dx) = \langle j, \nu \rangle,$$

in which

$$j := \int_{\mathbf{R}^3} \mu(dx) (y - p) \times \dot{y} \quad (2.6)$$

is the *moment of momentum about the point of contact* p . Here we have used that $\dot{A}x + \dot{a} = \dot{y}$, $A(x - s) = (y - a) - (p - a) = y - p$, and $\langle \dot{y}, \nu \times (y - p) \rangle = \langle (y - p) \times \dot{y}, \nu \rangle$.

We now turn to the case of Chaplygin's sphere [9], where the surface S of the body in the reference position is a sphere, the center of mass is at the center of S , and the body is rolling without slipping on a horizontal plane P . We will take the origin of the reference frame at the center of mass = the center of S . If r denotes the radius of S , and we take the plane P at height $-r$, then the condition that $A(S) + a$ is lying on top of P corresponds to the condition that the third (vertical) component of a is equal to zero. The point of contact then is equal to $p = a - r e_3$ if e_3 denotes the third standard basis vector, and the corresponding point on S , in body coordinates, is equal to

$$s = -r A^{-1} e_3. \quad (2.7)$$

The condition (2.2) therefore is equivalent to

$$\tilde{a} = r \nu \times e_3, \quad (2.8)$$

where we have also used (2.5).

From this moment on, we keep the infinitesimal rotation vector $\nu \in \mathbf{R}^3$ constant. Then (2.8) implies that \tilde{a} is a constant horizontal vector. If we use the *left* trivialization of the tangent bundle of $\text{SO}(3)$, corresponding to assigning to $\dot{A} \in T_A \text{SO}(3)$ the infinitesimal rotation given by

$$\dot{A}z = A(\omega \times z), \quad z \in \mathbf{R}^3 \quad (2.9)$$

for some vector $\omega \in \mathbf{R}^3$, then the tangent lift of the vector field \tilde{A} such that (2.5) with a constant ν does not effect ω . Because (2.8) implies that the tangent lift of \tilde{a} does not effect \dot{a} either, the conclusion is that condition ii) of Lemma 1.1 holds if we take $L = T$. Note that for Chaplygin's sphere the center of mass remains at the same height, which means that the gravitational potential energy is constant, and therefore can be disregarded. We have arrived at the conclusion that

Proposition 2.1 *For Chaplygin's sphere, the moment of momentum about the point of contact is a constant of motion.*

The kinetic energy of the rigid body is given by

$$T = \frac{1}{2} \langle I \omega, \omega \rangle + \frac{1}{2} m \langle \dot{a}, \dot{a} \rangle, \quad (2.10)$$

where we have used the left trivialization (2.9) of the tangent bundle of $\text{SO}(3)$. Here I denotes the moment of inertia tensor, which is given by a positive definite symmetric matrix, and m denotes the total mass of the body.

In this notation \tilde{A} , given by (2.5), corresponds to

$$\tilde{\omega} = A^{-1} \nu, \quad (2.11)$$

because

$$\tilde{A}z = \nu \times Az = A(A^{-1}\nu \times z).$$

It follows that

$$\begin{aligned} \langle j, \nu \rangle &= j^{(\tilde{A}, \tilde{a})} = \langle I \omega, A^{-1} \nu \rangle + m \langle \dot{a}, \tilde{a} \rangle \\ &= \langle A I \omega, \nu \rangle + m r^2 \langle A \omega \times e_3, \nu \times e_3 \rangle \\ &= \langle A I \omega + m r^2 e_3 \times (A \omega \times e_3), \nu \rangle, \end{aligned}$$

where we have used that (2.1), (2.7) and (2.9) imply that

$$\dot{a} = r A (\omega \times A^{-1} e_3) = r A \omega \times e_3$$

and δ is given by (2.8). With the notation

$$u := A^{-1} e_3 \quad (2.12)$$

this leads to the formula

$$j = A (I \omega + m r^2 u \times (\omega \times u)) \quad (2.13)$$

for the moment of momentum about the point of contact. Note that $\langle u, u \rangle = 1$ and therefore

$$u \times (\omega \times u) = \omega - \langle u, \omega \rangle u, \quad (2.14)$$

which is the orthogonal projection of ω onto the plane which is orthogonal to u . Also note that u has the concrete interpretation that $-r u$ is equal to the point of contact on the surface of the sphere, in body coordinates, cf. (2.7).

In order to simplify the notation somewhat, we write

$$\rho := m r^2, \quad (2.15)$$

and define the symmetric linear mapping $I_{\rho, u} : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ by

$$I_{\rho, u}(\omega) := (I + \rho) \omega - \rho \langle u, \omega \rangle u, \quad (2.16)$$

which is equal to I plus ρ times the orthogonal projection to the plane orthogonal to u . With these notations, we have that

$$j = A I_{\rho, u} \omega. \quad (2.17)$$

Write

$$J := (I + \rho)^{-1}. \quad (2.18)$$

If $I_{\rho, u} \omega = \nu$ then $\omega = J(\nu + \theta u)$ for some $\theta \in \mathbf{R}$, which moreover has to satisfy the equation

$$\nu = \nu + \theta u - \rho \langle u, J(\nu + \theta u) \rangle u,$$

which holds if and only if

$$\theta (1 - \rho \langle u, J u \rangle) - \rho \langle u, J \nu \rangle = 0.$$

This leads to the conclusion that the symmetric linear mapping $I_{\rho, u}$ is invertible, with inverse given by

$$I_{\rho, u}^{-1}(\nu) = J \nu + \frac{\rho \langle u, J \nu \rangle}{1 - \rho \langle u, J u \rangle} J u. \quad (2.19)$$

If we fix the constant of motion j , then we can use the equation

$$\omega = I_{\rho, u}^{-1} A^{-1} j \quad (2.20)$$

in order to express ω in terms of A , where we note that u is given in terms of A by means of (2.12). In this way ω can be eliminated from the equations of motion.

2.1 Chaplygin

In [9, §1], Proposition 2.1 is stated “... as a consequence of a generalized theorem of areas”, with a reference to [8].

In [8, §6] the case of [9] appears as the limit when the radius of the big sphere in which the small sphere rolls tends to infinity.

3 The Equations and Constants of Motion

3.1 The Equations of Motion

The equations of motion are

$$\frac{dp}{dt} = r A (\omega \times u) = r (A \omega) \times e_3, \quad (3.1)$$

$$\frac{dA}{dt} = A \circ \omega_{\text{op}}, \quad (3.2)$$

$$\frac{du}{dt} = u \times \omega, \quad u(t) := A(t)^{-1} e_3, \quad (3.3)$$

$$\frac{d}{dt} I \omega - I \omega \times \omega = m r^2 \left(\left\langle u, \frac{d\omega}{dt} \right\rangle u - \frac{d\omega}{dt} \right). \quad (3.4)$$

Here, as in Section 2, $x \mapsto A x + a$, $x, a \in \mathbf{R}^3$, $A \in \text{SO}(3)$ is the rigid motion which is applied to the body in the reference position, with the center of mass at the origin, and $p := A \tilde{s}(u) + a = a - r e_3$ denotes the point of contact between the sphere and the horizontal plane. Furthermore ω_{op} denotes the antisymmetric linear mapping $\nu \mapsto \omega \times \nu : \mathbf{R}^3 \rightarrow \mathbf{R}^3$. The equation (3.2) expresses that $\omega \in \mathbf{R}^3$ can be viewed as a rotational velocity vector.

The full system (3.1), (3.2), (3.4) is defined in the eight dimensional (p, A, ω) -space $\mathbf{R}^2 \times \text{SO}(3) \times \mathbf{R}^3$. The equations (3.2), (3.4) for

$$(A, \omega) \in \text{SO}(3) \times \mathbf{R}^3 \simeq \text{T}(\text{SO}(3))$$

are the equations for the *rotational motion*, the system obtained by *reduction by the horizontal translation group*. The equations (3.3), (3.4) for

$$(u, \omega) \in \text{S}^2 \times \mathbf{R}^3$$

represent the *reduction of the system by the left action of the horizontal motion group* $\text{E}(2)$.

With the notations (2.15) and (2.16), the equation (3.4) takes the form

$$I_{\rho, u} \frac{d\omega}{dt} = (I \omega) \times \omega. \quad (3.5)$$

Combining (2.16) with (3.3), we obtain that

$$\begin{aligned} \frac{dI_{\rho, u}}{dt} \omega &= -\rho \left(\left\langle \omega, \frac{du}{dt} \right\rangle u + \left\langle \omega, u \right\rangle \frac{du}{dt} \right) \\ &= -\rho (\langle \omega, u \times \omega \rangle u + \langle \omega, u \rangle u \times \omega) = -\rho \langle \omega, u \rangle u \times \omega, \end{aligned}$$

and therefore

$$\frac{d}{dt} (I_{\rho, u} \omega) = \frac{dI_{\rho, u}}{dt} \omega + I_{\rho, u} \frac{d\omega}{dt} = -\rho \langle \omega, u \rangle u \times \omega + (I \omega) \times \omega,$$

hence

$$\frac{d}{dt} (I_{\rho, u} \omega) = (I_{\rho, u} \omega) \times \omega, \quad (3.6)$$

because $\rho \omega \times \omega = 0$.

3.2 The Constants of Motion

In general the total energy is a constant of motion, when the equations of motions have been obtained as a consequence of the principle of d'Alembert, cf. (1.1). Because in our case the potential energy $m g r$ is a constant, it follows that the total kinetic energy

$$T = \frac{1}{2} \langle I \omega, \omega \rangle + \frac{1}{2} m r^2 \langle u \times \omega, u \times \omega \rangle = \frac{1}{2} \langle I_{\rho, u} \omega, \omega \rangle, \quad (3.7)$$

cf. (2.16), is a constant of motion. This can also be verified directly from (3.5) and (3.6), because

$$\langle I_{\rho, u} \omega, \frac{d\omega}{dt} \rangle = \langle \omega, I_{\rho, u} \frac{d\omega}{dt} \rangle = \langle \omega, (I \omega) \times \omega \rangle = 0$$

and therefore also

$$\frac{dT}{dt} = \left\langle \frac{d}{dt} (I_{\rho, u} \omega), \omega \right\rangle = \langle (I_{\rho, u} \omega) \times \omega, \omega \rangle = 0.$$

On the other hand, combination of (3.2) with (3.6) yields that

$$\begin{aligned} \frac{d}{dt} (A I_{\rho, u} \omega) &= \frac{dA}{dt} I_{\rho, u} \omega + A \frac{d}{dt} (I_{\rho, u} \omega) \\ &= A (\omega \times (I_{\rho, u} \omega)) + A ((I_{\rho, u} \omega) \times \omega) = 0. \end{aligned}$$

In this way we have verified again that the vector $A I_{\rho, u} \omega$, which according to (2.17) is equal to the moment j of the momentum around the point of contact, is a constant of motion.

3.3 A Pair of Vectors

If j is not vertical, then the rotation A is determined by the pair of vectors

$$u := A^{-1} e_3 \quad \text{and} \quad v := A^{-1} j. \quad (3.8)$$

More precisely, in this case the mapping $A \mapsto (u, v)$ is a diffeomorphism from $\text{SO}(3)$ onto the smooth algebraic submanifold of \mathbf{R}^6 , which consists of the $(u, v) \in \mathbf{R}^3 \times \mathbf{R}^3$ such that

$$\langle u, u \rangle = 1, \quad \langle u, v \rangle = j_3, \quad \langle v, v \rangle = \|j\|^2. \quad (3.9)$$

The equations of motion for the rotational motion are given in these coordinates by

$$\frac{du}{dt} = u \times \omega \quad \text{and} \quad \frac{dv}{dt} = v \times \omega, \quad (3.10)$$

in which ω is determined in terms of u and v by the equation

$$\omega = \omega(u, v) = I_{\rho, u}^{-1} v = J v + \frac{\rho \langle u, J v \rangle}{1 - \rho \langle u, J u \rangle} J u, \quad (3.11)$$

which in view of (3.8) is equivalent to (2.17). Here we have used the equation (2.19) in order to write ω even more explicitly as a function of u and v .

In view of (3.7), the kinetic energy can be expressed in terms of ω and v as

$$T = \frac{1}{2} \langle v, \omega \rangle, \quad (3.12)$$

which in view of (3.11) and (2.19) can be written in the form

$$T = \frac{1}{2} \langle v, Jv \rangle + \frac{1}{2} \frac{\rho \langle u, Jv \rangle^2}{1 - \rho \langle u, Ju \rangle}. \quad (3.13)$$

Later it will turn out to be convenient to write the kinetic energy equation in the form

$$f(u, v) := Y(u, v)^2 - X(u) Z(v) = 0, \quad (3.14)$$

in which

$$X(u) := \rho^{-1} - \langle u, Ju \rangle, \quad (3.15)$$

$$Y(u, v) := \langle u, Jv \rangle, \quad \text{and} \quad (3.16)$$

$$Z(v) := 2T - \langle v, Jv \rangle. \quad (3.17)$$

Note that $f(u, v)$ is a polynomial of degree four, but of degree two in each of the variables u and v separately.

3.4 The Left SO(2) Action

If R is a rotation about the vertical axis, then its action from the left sends A and \dot{A} to RA and $R\dot{A}$, respectively. It therefore leaves ω and u invariant and sends j to Rj . Note that the action of the group SO(2) of the rotations in \mathbf{R}^3 about the vertical axis is free on the set \mathcal{J}' of $j \in \mathbf{R}^3$ which are not equal to a multiple of e_3 . In \mathcal{J}' , the SO(2)-orbits are equal to the level curves of the functions $F(j) = \langle j, e_3 \rangle = j_3$ and $G(j) = \langle j, j \rangle = \|j\|^2$, where $j \in \mathcal{J}'$ corresponds to the condition that $F(j)^2 < G(j)$. Substituting (2.17) we obtain the constants of motion

$$j_3 = \langle j, e_3 \rangle = \langle I_{\rho, u} \omega, u \rangle = \langle I \omega, u \rangle \quad (3.18)$$

and

$$\|j\|^2 = \langle j, j \rangle = \langle I_{\rho, u} \omega, I_{\rho, u} \omega \rangle \quad (3.19)$$

for the left E(2)-reduced system for $(u, \omega) \in S^2 \times \mathbf{R}^3$.

Let

$$\pi : (A, \omega) \mapsto (u, \omega) = (A^{-1} e_3, \omega)$$

denote the projection from the phase space $T(\text{SO}(3)) \simeq \text{SO}(3) \times \mathbf{R}^3$ of the rotational motion onto the phase space $S^2 \times \mathbf{R}^3$ of the left E(2)-reduced system, which maps each left SO(2)-orbit to a point. The fact that the action of SO(2) on \mathcal{J}' is free implies that π is a diffeomorphism from the submanifold of $\text{SO}(3) \times \mathbf{R}^3$ determined by the equation $A I_{\rho, u} \omega = j$ onto the submanifold of $S^2 \times \mathbf{R}^3$ determined by the equations (3.18) and (3.19), where each of these submanifolds is invariant under motion of the system. The left SO(2)-invariance of the system means that π intertwines the rotational motion with the flow of the E(2)-reduced system.

3.5 Chaplygin

In the left column of the the following table we list the variables and some formulas which appear in Chaplygin [9, §2], with our corresponding notations in the right column. It is assumed that the moment of inertia tensor I is in diagonal form, in accordance to the “principal axes of inertia attached to the sphere” of Chaplygin [9, §2].

Chaplygin [9, §2]	our notation
(p, q, r)	ω
(u, v, w)	$A^{-1} \frac{dp}{dt}$
(P, Q, R)	$A^{-1} j = v$
$(\gamma, \gamma', \gamma'')$	$A^{-1} e_3 = u$
m	m
ρ	r
$D = m \rho^2$	$\rho = m r^2$
(L, M, N)	diagonal of I
(A, B, C)	diagonal of $I + \rho = J^{-1}$
(1)	(3.1)
(2)	(3.11)
ω in (3)	$-\langle \omega, u \rangle$
X	$X(u)$ in (3.15)
Y	$Y(u, v)$ in (3.16)
(6)	$(I + \rho) \omega = v + \frac{Y}{X} u, \quad \rho \langle \omega, u \rangle = \frac{Y}{X}$
(7)	(3.10) and (3.3)
n	$\ j\ ^2 = \langle j, j \rangle = \langle v, v \rangle$
h	$j_3 = \langle j, e_3 \rangle = \langle v, u \rangle$
l	$2T = \langle v, \omega \rangle$, cf. (3.12)
Z	$Z(v)$ in (3.17)
(10)	(3.14)

The only comment of Chaplygin to his formulas (1) and (2) consists of the preceding sentence “We easily find ...”.

No explicit notation has been introduced in Chaplygin [9, §2] for the rotation A . However, when Chaplygin said “(7) are the equations of motions of the sphere”, it is clear that he meant our Subsection 3.3.

The constants of motion of the $E(2)$ -reduced system are Chaplygin’s n , h and l , which correspond to our $\|j\|^2$, j_3 and $2T$, respectively.

4 The Level Surfaces of the Constants of Motion

4.1 Fixing the Moment

The system of equations (3.2), (3.4) in the (A, ω) -space $SO(3) \times \mathbf{R}^3$ describes the rotational motion of Chaplygin’s ball. It is equal to the system which is obtained by ignoring the

equation (3.1) for the motion of the point of contact (or the center of gravity), which is the same as the \mathbf{R}^2 -reduced system, obtained by working modulo the symmetry group of the horizontal translations $(A, \omega, a) \mapsto (A, \omega, a + b)$, where $b \in \mathbf{R}^2$ is viewed as a horizontal vector in \mathbf{R}^3 . The constants of motion, viewed as functions of (A, ω, a) in the phase space $\text{SO}(3) \times \mathbf{R}^3 \times \mathbf{R}^2$, do not depend on the horizontal translations a , and therefore will be considered as functions of (A, ω) in the phase space $\text{SO}(3) \times \mathbf{R}^3$ of the rotational motion.

As observed at after (2.20), the constant of motion j (= the moment of the momentum about the point of contact) can be used in order to eliminate ω from the equations of motion. In other words, $j : \text{SO}(3) \times \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is an analytic (rational) fibration, of which each fiber is equal to the graph of an analytic (rational) function ($\omega = \omega_j(A)$ a function of A), such that the projection $(A, \omega) \rightarrow A$ is an analytic (rational) diffeomorphism from the level set of j onto $\text{SO}(3)$.

We note that j is *not* invariant under the full symmetry group $E(2)$ (= the horizontal motion group) of the system. If R is a rotation around the vertical axis then it acts on the phase space by sending (A, ω, a) to (RA, ω, a) . It leaves $u = A^{-1}e_3$ invariant and we read off from (2.17) that it sends j to Rj . Therefore the level set is $E(2)$ -invariant if and only if the level j is vertical. If j is not vertical and R is a non-trivial rotation around the vertical axis, then the action of R on the phase space sends the level set at the level j to the different (disjoint) level set at the different level Rj .

The level set of all the constants of motion j and T together is diffeomorphic to the level set in $\text{SO}(3)$ of the function T_j on $\text{SO}(3)$, defined by $T_j(A) = T(A, \omega)$ when $AI_{\rho,u}\omega = j$. It follows from (3.7) that

$$T_j(A) = \frac{1}{2} \langle I_{\rho,u} \omega, \omega \rangle = \frac{1}{2} \langle j, A\omega \rangle = \frac{1}{2} \langle j, AI_{\rho,u}^{-1}A^{-1}j \rangle. \quad (4.1)$$

If $j = 0$ then, $\omega \equiv 0$, $\frac{dA}{dt} \equiv 0$, $T_j \equiv 0$, and Chaplygin's sphere is at rest. We will exclude this rather trivial case in the remainder of our discussions. The vector field of the motion in $\text{SO}(3)$ is given by (3.2), where $\omega = \omega_j(A)$ is given by (2.17). If $j \neq 0$ then $\omega_j(A) \neq 0$ and the vector field on $\text{SO}(3)$ has no zeros. It may also be observed that replacing j by cj with a constant c leads to replacing ω by $c\omega$, multiplying the vector field on $\text{SO}(3)$ by c , whereas $T_{cj} = c^2 T_j$. The solutions of the equations of motion are changed only by a rescaling of time by the constant factor c .

If T is a regular value of T_j , then the level set $\{A \in \text{SO}(3) \mid T_j(A) = T\}$ is a smooth (algebraic) closed two-dimensional submanifold of $\text{SO}(3)$, compact because $\text{SO}(3)$ is compact. It is oriented by the area form Ω/dT_j , where Ω is a volume form on $\text{SO}(3)$. Let C be a connected component of a regular level set. We conclude that C is a compact connected oriented two-dimensional smooth (algebraic) manifold which carries a tangent vector field without zeros, which implies that the Euler characteristic of C is equal to zero. According to the classification of compact oriented surfaces, this in turn implies that C is diffeomorphic to the two-dimensional *torus* $\mathbf{R}^2/\mathbf{Z}^2$. The considerations below will lead to a much more detailed description of the level sets, from which the conclusion that the regular ones consist of tori can be obtained without using the just mentioned facts from differential topology.

4.2 The Critical Points of the Energy

For any vector $\nu \in \mathbf{R}^3$, let $R_\nu(A) \in T_A \text{SO}(3)$ be the tangent vector of $\text{SO}(3)$ at $A \in \text{SO}(3)$ which is given by

$$R_\nu(A) := A \circ \nu_{\text{op}}. \quad (4.2)$$

In order to determine the derivative $R_\nu \omega_j$ of the vector-valued function ω_j on $\text{SO}(3)$ in the direction of the vector field R_ν , we begin with the observation that (2.17) implies that

$$0 = R_\nu (A I_{\rho,u} \omega_j) = A (\nu \times I_{\rho,u} \omega_j + R_\nu I_{\rho,u} \omega_j).$$

Furthermore,

$$R_\nu u = R_\nu A^{-1} e_3 - \nu \times A^{-1} e_3 = -\nu \times u,$$

and therefore it follows from (2.16) that

$$R_\nu I_{\rho,u} \omega_j = \rho (\langle \omega_j, \nu \times u \rangle u + \langle \omega_j, u \rangle \nu \times u) + I_{\rho,u} R_\nu \omega_j.$$

Therefore $R_\nu \omega_j$ is determined by the equation

$$0 = \nu \times ((I + \rho) \omega_j) + \rho \langle \omega_j, \nu \times u \rangle u + I_{\rho,u} R_\nu \omega_j. \quad (4.3)$$

Because $I_{\rho,u}$ is symmetric, we have that

$$\frac{1}{2} \langle I_{\rho,u} R_\nu \omega_j, \omega_j \rangle + \frac{1}{2} \langle I_{\rho,u} \omega_j, R_\nu \omega_j \rangle = \langle I_{\rho,u} R_\nu \omega_j, \omega_j \rangle.$$

The derivative of T_j , cf. (3.7), in the direction of R_ν therefore is equal to

$$\begin{aligned} R_\nu T_j &= \frac{1}{2} \langle (R_\nu I_{\rho,u}) \omega_j, \omega_j \rangle + \langle I_{\rho,u} R_\nu \omega_j, \omega_j \rangle \\ &= \frac{1}{2} \langle \langle \omega_j, \nu \times u \rangle u + \langle \omega_j, u \rangle \nu \times u \\ &\quad - \langle \nu \times ((I + \rho) \omega_j) + \rho \langle \omega_j, \nu \times u \rangle u, \omega_j \rangle, \end{aligned}$$

from which we obtain that

$$R_\nu T_j = \langle \omega_j \times (I + \rho) \omega_j, \nu \rangle = \langle \omega_j \times I \omega_j, \nu \rangle. \quad (4.4)$$

Let Σ_j denote the set of critical points of T_j . It follows from (4.4) that $A \in \Sigma_j$ if and only if $\omega = \omega_j(A)$ satisfies $\omega \times I \omega = 0$. In view of (3.4) this is equivalent to $\frac{d\omega}{dt} = 0$. Because the function T_j is invariant under the motion, the R_{ω_j} -flow on $\text{SO}(3)$, Σ_j is invariant under the motion. Therefore, if $A(0) \in \Sigma_j$ then we have for every t that $A(t) \in \Sigma_j$, which implies that $\frac{d\omega_j(A(t))}{dt} = 0$ for every t and therefore $\omega = \omega_j(A(t))$ is a constant. It follows then from (3.2) that $A(t) = A(0) \circ e^{t\omega}$ describes a circle, which we will call a *critical circle*.

We have $\omega \times I \omega = 0$ if and only if $I \omega = \iota \omega$, which means that ω is an eigenvector of the moment of inertia tensor I , with eigenvalue ι equal to one of the principal inertial moments I_1, I_2 or I_3 . If this is the case, it then follows from (2.17) that

$$j = (\iota + \rho) A \omega - \rho \langle A \omega, e_3 \rangle e_3, \quad (4.5)$$

Taking the inner product with e_3 we obtain that

$$j_3 = \iota \langle A\omega, e_3 \rangle, \quad (4.6)$$

which can be inserted into (4.5) in order to yield that

$$A\omega = \frac{1}{\iota + \rho} \left(j + \frac{\rho j_3}{\iota} e_3 \right). \quad (4.7)$$

From (4.7) we obtain that

$$\|\omega\|^2 = \frac{1}{(\iota + \rho)^2} \left(\|j\|^2 + 2\frac{\rho j_3^2}{\iota} + \left(\frac{\rho j_3}{\iota} \right)^2 \right), \quad (4.8)$$

and combining (4.7) with (4.1) we obtain that the critical level is equal to

$$T_{\text{crit}} = \frac{1}{2} \frac{1}{\iota + \rho} \left(\|j\|^2 + \frac{\rho j_3^2}{\iota} \right). \quad (4.9)$$

Note that the right hand side of (4.9) is a monotonously decreasing function of ι , which implies that if the principal inertial moments are taken in increasing order, then the corresponding critical levels of the kinetic energy appear in decreasing order.

It follows from (4.7) and (3.1) that *the point of contact $p(t)$ moves along a straight line, with constant velocity equal to*

$$\frac{dp}{dt} = \frac{r}{\iota + \rho} j \times e_3. \quad (4.10)$$

We note that for the uniformly rolling sphere the axis of rotation $\mathbf{R} A\omega$ need not be horizontal as one might expect. It follows from (4.6) that it is horizontal if and only if the vector j is horizontal.

The point of contact is at rest if and only if the moment of momentum is vertical (j is equal to a multiple of e_3), which according to (4.7) corresponds to the case that the vector $A\omega$ is vertical. In this case the sphere is spinning around the vertical axis, which then coincides with an inertial axis.

Let $E_{j,\iota}$ denote the set of $\omega \in \ker(I - \iota)$ such that (4.8) holds, and let $\Sigma_{j,\iota}$ denote the set of $A \in \text{SO}(3)$ such that (4.7) holds for some $\omega \in E_{j,\iota}$. Then Σ_j is a smooth algebraic circle bundle over the smooth algebraic manifold $E_{j,\iota}$ and therefore is an smooth algebraic submanifold of $\text{SO}(3)$. For different values of the eigenvalue ι of I , the sets $\Sigma_{j,\iota}$ are disjoint and Σ_j is equal to the union of the $\Sigma_{j,\iota}$, where ι runs over the principal inertial moments.

If ι is a simple eigenvalue of I then $E_{j,\iota}$ consist of two opposite ($\omega \leftrightarrow -\omega$) eigenvectors of I for the eigenvalue ι , and $\Sigma_{j,\iota}$ consists of two disjoint critical circles. Note that the function T_j (and also the equation of motion) is invariant under a transformation $A \mapsto AR$, where $R \in \text{SO}(3)$ commutes with I . There exists such R which maps ω to $-\omega$ and for each such R the mapping $A \mapsto AR$ interchanges the two critical circles in $\Sigma_{j,\iota}$. In the generic case that

all the principal inertial moments I_1, I_2, I_3 are different, we obtain six critical circles, two for every choice of $\iota = I_1, I_2, I_3$.

The case of two equal principal moments of inertia $I_1 = I_2 \neq I_3$ is that of a body of revolution with surface equal to a sphere and center of mass at the center of the sphere. This is the example of *Routh's sphere with center of mass at the center of the sphere*, or *Bobylev's sphere*, which will be discussed in some more detail in Section 6. If $\iota = I_1 = I_2 \neq I_3$ then $E_{j,\iota}$ is a circle in $\ker(I - \iota)$ and $\Sigma_{j,\iota}$ is a two-dimensional torus in $\text{SO}(3)$. As discussed before, Σ_{j,I_3} consists of two critical circles.

If all the principal moments of inertia are equal, $\iota = I_1 = I_2 = I_3$, or equivalently I is equal to ι times the identity, then $\Sigma_j = \text{SO}(3)$, the function T_j is constant, and all solutions of the equations of motion are of the form $A(t) = A(0) \circ e^{t\omega}$ with a constant vector ω .

We will now verify that *each $\Sigma_{j,\iota}$ is a nondegenerate critical manifold* of T_j in the sense of Bott [7], which means that for each $A \in \Sigma_{j,\iota}$ the null space of the Hessian $T_j''(A)$ of T_j at A is equal to the tangent space $T_A \Sigma_{j,\iota}$ of $\Sigma_{j,\iota}$ at A . Because always $T_A \Sigma_{j,\iota} \subset \ker T_j''(A)$, we only need to verify that the dimension of the null space of the Hessian is at most equal to the dimension of the critical submanifold. This follows from

Lemma 4.1 *Let $I\omega_j(A) = \iota\omega_j(A)$ and $R_\nu(A) \in \ker T_j''(A)$. If ι is a simple eigenvalue of I , then ν is a multiple of $\omega_j(A)$. If ι is a double eigenvalue of I and α is a nonzero vector which is orthogonal to $\ker(I - \iota)$, then ν is a linear combination of $\omega_j(A)$ and α .*

Proof It follows from (4.4) that $R_\nu(A) \in \ker T_j''(A)$ if and only if, at the point $A \in \text{SO}(3)$,

$$0 = R_\nu(I\omega_j \times \omega_j) = I R_\nu \omega_j \times \omega_j + I\omega_j \times R_\nu \omega_j = (I R_\nu \omega_j - \iota R_\nu \omega_j) \times \omega,$$

or $(I - \iota) R_\nu \omega_j$ is equal to a multiple of ω_j . This implies that $(I - \iota)^2 R_\nu \omega_j = 0$, which in turn implies that $(I - \iota) R_\nu \omega_j = 0$, because $I - \iota$ is a diagonal matrix.

Substituting $I\omega_j = \iota\omega_j$ and $I R_\nu \omega_j = \iota R_\nu \omega_j$ in (4.3), we obtain

$$\begin{aligned} 0 &= \nu \times ((\iota + \rho)\omega_j) + \rho \langle \omega_j, \nu \times u \rangle u + (\iota + \rho) R_\nu \omega_j - \rho \langle R_\nu \omega_j, \nu \times u \rangle u \\ &= (\iota + \rho)\theta - \rho \langle \theta, u \rangle u, \end{aligned}$$

in which $\theta := \nu \times \omega_j + R_\nu \omega_j$. It follows that $\theta = 0$.

If ι is a simple eigenvalue of I then $I R_\nu \omega_j = \iota R_\nu \omega_j$ implies that $R_\nu \omega_j$ is equal to a multiple of ω_j and it follows from $\theta = 0$ that $\nu \times \omega_j = 0$ and $R_\nu \omega_j = 0$, which in turn implies that ν is a multiple of ω_j .

If $\ker(I - \iota)$ is two-dimensional, then $\theta = 0$ implies that $0 = \langle \nu_j, \alpha \rangle = \langle \nu, \omega_j \times \alpha \rangle$. Because $\omega_j \in \ker(I - \iota)$, we have that α is orthogonal to ω_j and it follows that ν is a linear combination of $\omega_j(A)$ and α . \square

Because T_j is a continuous function on the compact set $\text{SO}(3)$, it attains its maximum and its minimum. The variational principle says that the points A where T_j attains its maximum (minimum) are critical points for T_j . Therefore, if $I_1 \leq I_2 \leq I_3$, then the maximum

(minimum) value of T_j is equal to the right hand side of (4.9), with $\iota = I_1$ ($\iota = I_3$). If $I_1 < I_2$ ($I_2 < I_3$), then $\Sigma_{j,\iota}$ consists of two critical circles and on a transversal two-dimensional manifold the function T_j has a nondegenerate maximum (minimum), the nearby level sets of which are small loops around the critical points. It follows that the level sets of T_j near $\Sigma_{j,\iota}$ consist of narrow tubes around the critical circles. This implies that the critical circles corresponding to $\iota = I_1 < I_2 \leq I_3$ or to $\iota = I_3 > I_2 \geq I_1$ are *stable* periodic solutions of the system.

Now assume that $I_1 < I_2 < I_3$. A Morse theoretic argument then yields that T_j has cannot have a local maximum or minimum at Σ_{j,I_2} . Note that the index of T_j'' , the number of negative eigenvalues of T_j'' , is constant along each of the two critical circles in Σ_{j,I_2} . If $R = \text{diag}(-1, -1, 1)$ or $R = \text{diag}(1, -1, -1)$, then $A \mapsto AR$ leaves T_j invariant and interchanges the two critical circles in Σ_{j,I_2} , which implies that the index of T_j'' is constant along Σ_{j,I_2} . Suppose that it is equal to two, which means that T_j has a local maximum at Σ_{j,I_2} . Let $\text{grad } T_j$ denote the gradient vector field of T_j with respect to a given Riemannian structure on $\text{SO}(3)$. Let γ^t denote the flow of $\text{grad } T_j$. Define S_1 and S_2 as the set of $A \in \text{SO}(3)$ such that, when $t \rightarrow \infty$, $\gamma^t(A)$ converges to Σ_{j,I_1} and Σ_{j,I_2} , respectively. S_1 and S_2 are nonvoid disjoint open subsets of $\text{SO}(3)$, with union equal to $\text{SO}(3) \setminus \Sigma_{j,I_3}$. The set $\text{SO}(3) \setminus \Sigma_{j,I_3}$ is connected, because Σ_{j,I_3} is a codimension two submanifold of the connected manifold $\text{SO}(3)$. This leads to a contradiction. In a similar way the assumption that T_j has a local maximum at Σ_{j,I_2} leads to a contradiction and the conclusion is that the index of T_j'' is equal to one along Σ_{j,I_2} , which means that transversally T_j has a saddle point behaviour.

Let P denote the linearization of the return map to a transversal plane (the Poincaré map) of the flow along the critical circles in Σ_{j,I_2} . It follows from Corollary 7.2 that $\det P = 1$. This implies that $|\text{trace } P| < 2$ if and only if P is conjugate to a nontrivial rotation, whereas $|\text{trace } P| > 2$ if and only if P is a hyperbolic map with real eigenvalues $\lambda, 1/\lambda$ such that $\lambda \neq \pm 1$, in which case the critical circles in Σ_{j,I_2} are linearly unstable. The saddle point behaviour of T_j near Σ_{j,I_2} (when $I_1 < I_2 < I_3$) excludes that P is conjugate to a nontrivial rotation. The number $\text{trace } P$ depends in a real analytic fashion on $j \in \mathbf{R}^3 \setminus \{0\}$. As we will see at the end of Subsection 5 below, if j is vertical then the critical circles in Σ_{j,I_2} are linearly unstable, which implies that $|\text{trace } P| > 2$ when j is vertical. It follows that the set N of $j \in \mathbf{R}^3 \setminus \{0\}$ such that $\text{trace } P = \pm 2$ is a proper closed analytic subvariety of $j \in \mathbf{R}^3 \setminus \{0\}$. Using the invariance of the equations of motion under the action $(A, \omega) \mapsto (RA, \omega)$, $j \mapsto Rj$ of the rotations R around the vertical axis and the homogeneity $(A, \omega) \mapsto (A, c\omega)$, $j \mapsto cj$, it follows that N , if not empty, is equal to the union of finitely many cones in \mathbf{R}^3 which are invariant under the rotations around the vertical axis. For all j in the complement of N , which is an open and dense subset of $\mathbf{R}^3 \setminus \{0\}$, the critical circles in Σ_{j,I_2} are linearly unstable. This in turn implies that *every critical circle with $I\omega = I_2\omega$ is unstable with respect to the flow in the full phase space.*

Conversely, at any critical circle which is linearly unstable the invariant function T_j must have transversal saddle point behaviour. Because the linearly unstable critical circles are dense, this leads to a proof that the index of T_j is equal to one along Σ_{j,I_2} without using

Morse theory.

Question 4.2 When $I_1 < I_2 < I_3$, is every critical circle with $I\omega = I_2\omega$ linearly unstable? \oslash

4.3 The Moment Mapping

It is also instructive to consider (j, T) as a mapping from the phase space $\text{SO}(3) \times \mathbf{R}^3$ of the rotational motion to $\mathbf{R}^3 \times \mathbf{R}$. The set of *singular points* of (j, T) , the set of points where the rank of the tangent map is less than four, is equal to the union of $\text{SO}(3) \times \{0\}$ and the set of all (A, ω) where $j \in \mathbf{R}^3 \setminus \{0\}$, $A \in \Sigma_j$ and $\omega = \omega_j(A)$. If $I_1 < I_2 < I_3$ then the latter set is a smooth conic closed submanifold of codimension two in $\text{SO}(3) \times (\mathbf{R}^3 \setminus \{0\})$.

Because of the scaling $(A, \omega) \mapsto (A, c\omega)$, which maps (j, T) to (cj, c^2T) , and because the case $T = 0$, when $\omega = 0$ and everything is at rest, is not very interesting, we restrict ourselves to an energy hypersurface where T is equal to a positive constant. (An isotopy argument as below shows that the energy hypersurface is diffeomorphic to the Cartesian product of $\text{SO}(3)$ with a two-dimensional sphere.) The moment of the momentum around the point of contact then defines a mapping j_T from the energy hypersurface to \mathbf{R}^3 . The singular points of j_T are the above singular points on the energy hypersurface.

According to (4.9), the set of singular values of j_T consists of the points $j \in \mathbf{R}^3$ such that

$$\frac{1}{2(\iota + \rho)} \left(\|j\|^2 + \frac{\rho}{\iota} j_3^2 \right) = T, \quad (4.11)$$

in which $\iota = I_1$, $\iota = I_2$ or $\iota = I_3$. If $I_1 < I_2 < I_3$, then this set is equal to the union of three disjoint ellipsoids with center at the origin and which are invariant under the rotations around the vertical axis. The inner and the outer one correspond to $\iota = I_1$ and $\iota = I_3$, respectively. The points in the phase space of the rotational motion which by j_T are mapped to these ellipsoids correspond to the extremal critical circles (the stable ones) of the functions T_j , and it follows that the inner and outer ellipsoids together form the boundary of the image of j_T . Therefore, the image of j_T is equal to the set of all $j \in \mathbf{R}^3$ such that (4.11) holds for some $\iota \in [I_1, I_3]$.

The unstable critical circles are mapped to the intermediate (interior) ellipsoid described by (4.11) with $\iota = I_2$. For the singular values j in this interior ellipsoid, the level sets are two-dimensional, with a singularity of normal crossing type along the two unstable critical circles in the level set.

For the regular values of j_T , the points j such that (4.11) holds for some ι such that $I_1 < \iota < I_2$ or $I_2 < \iota < I_3$, the level sets are smooth (algebraic) two-dimensional compact oriented submanifolds of the phase space for the rotational motion.

4.4 Isotopy of the Fibration

Following Cushman [11, p. 412], a smooth function on a compact manifold, for which the critical set consists of nondegenerate critical manifolds (possibly with varying dimensions),

will be called a *Bott-Morse function*. We will use an isotopy lemma for families of Bott-Morse functions, which should be well-known. However, because we did not find a reference in the literature, we include a proof.

Lemma 4.3 *Let M be a compact smooth manifold and f_ϵ a familie of smooth functions on M , depending smoothly on a real parameter ϵ . Furthermore assume that the set of critical points of f_ϵ consists of finitely many disjoint compact connected smooth submanifolds $C_{\epsilon,i}$, $1 \leq i \leq N$, depending smoothly on ϵ and such that $C_{\epsilon,i}$ is a nondegenerate critical manifold of $f_{\epsilon,i}$. On $C_{\epsilon,i}$ the function f_ϵ is constant, let $F_{\epsilon,i}$ be the value of f_ϵ on $C_{\epsilon,i}$. We finally assume that the ordering of the real numbers $F_{\epsilon,i}$, $1 \leq i \leq N$, does not change with varying ϵ . Under these assumptions there exist smooth diffeomorphisms ψ_ϵ and Φ_ϵ of \mathbf{R} and M respectively, such that $\psi_\epsilon \circ f_\epsilon \circ \Phi_\epsilon$ does not depend on ϵ . The ψ_ϵ can be chosen to be order-preserving.*

Proof The assumption that the ordering of the critical values does not change implies that there exists a family of order-preserving smooth diffeomorphisms ψ_ϵ of \mathbf{R} such that the real numbers $G_i := \psi_\epsilon(F_{\epsilon,i})$, $1 \leq i \leq N$ do not depend on ϵ . It is also quite easy to prove that there exists a smooth family of diffeomorphisms Ψ_ϵ of M , depending smoothly on ϵ , such that the manifolds $D_i := \Psi_\epsilon^{-1}(C_{\epsilon,i})$ do not depend on ϵ . The functions $g_\epsilon := \psi_\epsilon \circ f_\epsilon \circ \Psi_\epsilon$ have the same properties as f_ϵ , but now with the constant nondegenerate critical manifolds D_i on which g_ϵ has the constant critical values G_i .

We now follow the idea of the proof of Moser [29]. The condition for a smooth family of diffeomorphisms Ξ_ϵ of M , depending smoothly on ϵ , that $g_\epsilon \circ \Xi_\epsilon = \Xi_\epsilon^* g_\epsilon$ does not depend on ϵ , is equivalent to the condition that $0 = \frac{d}{d\epsilon} \Xi_\epsilon^* g_\epsilon = \Xi_\epsilon^* (X_\epsilon g_\epsilon + \frac{\partial g_\epsilon}{\partial \epsilon})$, or

$$X_\epsilon g_\epsilon + \frac{\partial g_\epsilon}{\partial \epsilon} = 0, \quad (4.12)$$

in which X_ϵ denotes the vector field on M defined by

$$\frac{\partial \Xi_\epsilon(x)}{\partial \epsilon} = X_\epsilon(\Xi_\epsilon(x)), \quad x \in M. \quad (4.13)$$

If the equation (4.12) for X_ϵ can be solved locally near every point of M , then a global solution can be obtained by means of a smooth partition of unity.

Near a noncritical point of g_ϵ we can use g_ϵ as one of the local coordinates and (4.12) then amounts to prescribing the corresponding component of the vector field X_ϵ .

If $x^{(0)} \in D_i$, then one can introduce a local coordinate system near $x^{(0)}$ in which $x^{(0)} = 0$ and $x \in D_i$ corresponds to $x_j = 0$ for $1 \leq j \leq c$, if c denotes the codimension of C_i in M . Writing $x = (y, z)$ with $y \in \mathbf{R}^c$, $z \in \mathbf{R}^d$, we view $g_\epsilon(y, z)$ as a family of functions of y , with ϵ and z as parameters. A second order Taylor expansion with respect to y at $y = 0$, in which the remainder term in integral form is absorbed into the second order term, yields that

$$g_\epsilon(y, z) = G_i + \frac{1}{2} \langle Q_\epsilon(y, z) y, y \rangle,$$

where $Q = Q_\epsilon(y, z)$ is a nondegenerate symmetric matrix, depending smoothly on all the variables. (This is also one of the steps in the proof of the Morse lemma with parameters

of Hörmander [20, Lemma 3.2.3].) If we take $X_\epsilon = (Y_\epsilon, 0)$ with $Y_\epsilon \in \mathbf{R}^c$, then the equation (4.12) for X_ϵ is equivalent to the equation

$$\langle Q_\epsilon(y, z) Y_\epsilon, y \rangle + \frac{1}{2} \left\langle \left(\frac{\partial Q_\epsilon(y, z)}{\partial y} Y_\epsilon \right) y, y \right\rangle + \frac{1}{2} \left\langle \frac{\partial Q_\epsilon(y, z)}{\partial \epsilon} y, y \right\rangle = 0$$

for Y_ϵ , which is satisfied if

$$Q_\epsilon(y, z) Y_\epsilon + \frac{1}{2} \left(\frac{\partial Q_\epsilon(y, z)}{\partial y} Y_\epsilon \right) y + \frac{1}{2} \frac{\partial Q_\epsilon(y, z)}{\partial \epsilon} y = 0.$$

For sufficiently small y the latter equation has a unique solution Y_ϵ which depends smoothly on y , z , and ϵ .

Piecing together the local solutions by means of a smooth partition of unity, we obtain a smooth vector field X_ϵ on M , depending smoothly on ϵ , such that (4.12) holds. Define $\Xi_\epsilon(x)$ as the solution of the ϵ -dependent ordinary differential equation (4.13), with initial condition $\Xi_0(x) = x$. Using the compactness of M we obtain that the Ξ_ϵ are globally defined smooth diffeomorphisms of M , depending smoothly on ϵ . Reading the paragraph preceding (4.13) backwards, we obtain that $g_\epsilon \circ \Xi_\epsilon$ does not depend on ϵ . This proves the lemma with $\Phi_\epsilon = \Psi_\epsilon \circ \Xi_\epsilon$. \square

In order to emphasize the dependence on ρ of the kinetic energy function on $\text{SO}(3)$, we now write $T_{\rho, j}$ instead of T_j . Applying Lemma 4.3 to $f_\epsilon = T_{\epsilon, j}$, we obtain that there exists an order-preserving smooth diffeomorphism ψ of \mathbf{R} and a smooth diffeomorphism Φ of $\text{SO}(3)$ such that $\psi \circ T_{\rho, j} \circ \Phi = T_{0, j}$.

We can make ψ and Φ to depend smoothly on j when j varies over the unit sphere in \mathbf{R}^3 . Extending the transformations by homogeneity for the scaling $(A, \omega) \mapsto (A, c\omega)$, one obtains a diffeomorphism Φ of $\text{SO}(3) \times (\mathbf{R}^3 \setminus \{0\})$, and a diffeomorphism Ψ of $(\mathbf{R}^3 \setminus \{0\}) \times \mathbf{R}$ of the form $(j, T) \mapsto (j, \psi(j, T))$, such that $\psi(cj, c^2 T) = c^2 \psi(j, T)$ for every $c > 0$, such that $\Psi \circ (j_\rho, T_\rho) \circ \Phi = (j_0, T_0)$. This implies that the smooth diffeomorphism Φ maps the whole fibration, together with its singularities, of the phase space defined by the constants of motion for $\rho = 0$ to the one for our given value of ρ .

If $\rho = 0$, then $I_{\rho, u} = I_{0, u} = I$. The equation of motion (3.4) then turns into Euler's equation of motion

$$\frac{d}{dt} I \omega = I \omega \times \omega \quad (4.14)$$

for the Euler top, and the constants of motion j of (2.17) and T of (4.1) are given by the familiar formulas

$$j = A I \omega \quad (4.15)$$

and

$$T_{0, j}(A) = \frac{1}{2} \langle I \omega, \omega \rangle = \frac{1}{2} \langle A^{-1} j, I^{-1} A^{-1} j \rangle \quad (4.16)$$

for the moment of momentum around the center of mass and the kinetic energy of the Euler top, respectively. For details about the Euler top, we refer to Cushman and Bates [12, Ch. III].

It follows from (4.16) that the kinetic energy $T_{0,j}$ of the Euler top is invariant under the circle action $A \mapsto e^{t j_{\text{op}}} \circ A$, the orbits of which are the fibers of the mapping $v_j : A \mapsto A^{-1} j$. Note that $v_j : \text{SO}(3) \rightarrow S_{\|j\|}$ is a smooth fibration of $\text{SO}(3)$ over the *Euler sphere* $S_{\|j\|}$, the sphere in \mathbf{R}^3 with center at the origin and radius equal to $\|j\|$. On the Euler sphere, the kinetic energy is equal to the restriction to $S_{\|j\|}$ of the quadratic form $v \mapsto \frac{1}{2} \langle v, I^{-1} v \rangle$, defined by the positive definite symmetric matrix I^{-1} .

The critical levels of $T_{0,j}$ are equal to $\frac{1}{2} \frac{\|j\|^2}{\iota}$, where $\iota = I_1, I_2$ or I_3 , and the regular values are the numbers in between the critical levels. On the Euler sphere each regular level set has two connected components, opposite to each other, each of which is a smooth closed curve, diffeomorphic to a circle. The preimages of these under the mapping v_j are circle bundles over these circles and therefore each regular level set in $\text{SO}(3)$ has two connected components, each of which is diffeomorphic to the two-dimensional torus.

For each extremal level the level sets consists of two critical circles surrounded by narrow tubes. For the intermediate critical level, the level set on $S_{\|j\|}$ consists of two opposite critical points. The complement of these in the level set has four connected component, each of which is a smooth curve running from one of the critical points to the opposite one. It follows that the level set in $\text{SO}(3)$ of the intermediate critical level contains two critical circles, the complement of which has four connected components each of which is a smooth cylinder running from one of the critical circles to the other.

For more details about the fibration in $\text{SO}(3) \times \mathbf{R}^3$ for the Euler top we refer to Cushman and Bates [12, Ch. III, Sec. 5]. The point of the isotopy lemma is that there exists a diffeomorphism Φ which sends the whole fibration with singularities for the Euler top to the one for Chaplygin's sphere for an arbitrary value of ρ . In particular all the qualitative statements about the level sets remain true. The regular level sets have two connected components, each of which is diffeomorphic to the two-dimensional torus. The level set of an intermediate critical level contains two critical circles, the complement of which has four connected components each of which is a smooth cylinder running from one of the critical circles to the other.

4.5 Chaplygin

In the beginning of [9, §6] Chaplygin gave a short description of the critical circles, but without relating these solutions to the points where the derivatives of the constants of motion are linearly dependent. He also stated that the ones corresponding to the extremal moments of inertia are stable and the ones corresponding to the intermediate moment of inertia are unstable, but without any proof.

The question of the smoothness of the level surface of the constant of motion, which is related to the question of the linear independence of their derivatives, does not occur in Chaplygin [9].

5 When the Moment is Vertical

Next to the critical circles, the solutions with *vertical moment* j of the momentum around the point of contact, $j = j_3 e_3$, form another interesting special family.

Note that the condition that j is vertical defines a smooth codimension two algebraic submanifold of $\text{SO}(3) \times (\mathbf{R}^3 \setminus \{0\}) \times \mathbf{R}^2$. If $I_1 < I_2 < I_3$ then also the set of singular points of the constants of motion, corresponding to the critical circles, is a smooth codimension two algebraic submanifold of $\text{SO}(3) \times (\mathbf{R}^3 \setminus \{0\}) \times \mathbf{R}^2$. The intersection of these submanifolds of special motions consists of the rotations of the sphere around a vertical axis which is equal to an axis of inertia, during which the point of contact is at rest. These motions define a submanifold of codimension four in $\text{SO}(3) \times (\mathbf{R}^3 \setminus \{0\}) \times \mathbf{R}^2$.

5.1 Invariance under Rotations about the Vertical Axis

If j is vertical, then j is invariant under the group of rotations around the vertical axis, which is the action of the $E(2)$ -symmetry group, the horizontal motion group, on j . This implies that the level set of j is invariant under the transformations $(A, \omega) \mapsto (RA, \omega)$ with $R \in \text{SO}(2)$. The orbits of this action are equal to the fibers of the projection $(A, \omega) \mapsto (u, \omega)$, $u = A^{-1} e_3$, where the space $S \times \mathbf{R}^3$ of the $(u, \omega) \in \mathbf{R}^3 \times \mathbf{R}^3$ such that $\|u\| = 1$ is viewed as the phase space for the $E(2)$ -reduced system. Because T is $E(2)$ -invariant, it follows that the function T_j on $\text{SO}(3)$, which represents T on the j -level set, is $\text{SO}(2)$ -invariant, a fact which can also be deduced directly from (4.1). In particular the two-dimensional regular level sets of T_j are $\text{SO}(2)$ -invariant, and therefore are mapped by the projection $A \mapsto (u, \omega)$, $u = A^{-1} e_3$, $\omega = \omega_j(A)$, to smooth compact one-dimensional algebraic submanifolds of $S \times \mathbf{R}^3$. Each connected component of the regular level set in the (u, ω) -space therefore will be a closed curve, which implies that the motion in the $E(2)$ -reduced phase space is *periodic*.

As a consequence, we can apply the reconstruction technique in Hermans [18, Sec. 3.2] in order to obtain information about the flow in the full (A, ω, a) -phase space $\text{SO}(3) \times \mathbf{R}^3 \times \mathbf{R}^2$ of the rolling body. There it is assumed that the symmetry group is compact, but the only thing which is needed is that the centralizer in the group of the shift element is a torus. Now the centralizer of the element $(B, b) \in E(2) \simeq \text{SO}(2) \times \mathbf{R}^2$ is a circle subgroup of $E(2)$ when $B \neq I$, or B is a nontrivial rotation around the vertical axis, whereas it is equal to the translation subgroup \mathbf{R}^2 if $B = I$. Because the $E(2)$ -reduced phase space $S^2 \times \mathbf{R}^3$ is equal to the $\text{SO}(2)$ -reduced phase space of the \mathbf{R}^2 -reduced phase space $\text{SO}(3) \times \mathbf{R}^3$, we obtain the following conclusion.

Proposition 5.1 *Let j be vertical. Each solution on a regular level set in the (u, ω) -space, the $E(2)$ -reduced phase space, is periodic. The corresponding rotational motion in the (A, ω) -space is quasiperiodic on an analytic two-dimensional torus, depending analytically on the parameters j and T .*

For each periodic solution in the (u, ω) -space such that the corresponding solution in the (A, ω) -space is not periodic with the same period, the motion in the full phase space $\text{SO}(3) \times \mathbf{R}^3 \times \mathbf{R}^2$ is quasiperiodic on an analytic two-dimensional torus, depending analytically on j

and T . In particular the point of contact $p(t)$ remains in a bounded subset of the plane in this case.

If the rotational motion, the motion in the (A, ω) -space, is periodic with the same minimal period as the motion in the (u, ω) -space, then the translational motion, the motion of the point of contact $p(t)$, is equal to the superposition of a straight line motion with constant speed and a periodic motion with the same period as the motion in the (u, ω) -space.

Some more information about the periodic solutions mentioned in Proposition 5.1 is given in Proposition 5.2 below. Proposition 5.3 implies that for at least half of these periodic rotational motions the motion of the point of contact $p(t)$ is actually periodic, with the same period.

For more details on the behaviour of the sphere when the moment is vertical, see Kilin [22, Sec. 3.3], which also contains computer pictures of orbits of the point of contact.

5.2 Fourier Series

In order to appreciate the statements in Proposition 5.1, we recall what it means that the motion in a suitable open subset M of the phase space is quasiperiodic on analytic r -dimensional tori, with analytic dependence on s parameters, varying in some open subset E of \mathbf{R}^s . It means that there exists an analytic diffeomorphism Φ from $(\mathbf{R}^r/\mathbf{Z}^r) \times E$ to M and an analytic function $\nu : E \rightarrow \mathbf{R}^r$, such that the pull-back $w = \Phi^* v$ of the velocity field v on M is of the form $w(x, \epsilon) = (\nu(\epsilon), 0)$, $x \in \mathbf{R}^r/\mathbf{Z}^r$, $\epsilon \in E$. This implies that the solution curves γ in M are of the form $\gamma(t) = \Phi(x_0 + t\nu(\epsilon), \epsilon)$, $t \in \mathbf{R}$. In our case $r = 2$.

If we apply this to the rotational motion, then it follows that the right hand side $r(A\omega) \times e_3$ in (3.1) is of the form $\dot{p}(x_0 + t\nu(\epsilon), \epsilon)$, where \dot{p} is an analytic mapping from $(\mathbf{R}^2/\mathbf{Z}^2) \times E$ to the horizontal plane \mathbf{R}^2 . Using Fourier expansion it follows that we can write

$$\dot{p}(x_0 + t\nu(\epsilon), \epsilon) = \sum_{k \in \mathbf{Z}^2} c_k(\epsilon) e^{2\pi i \langle x_0 + t\nu(\epsilon), k \rangle}, \quad (5.1)$$

in which the Fourier coefficients $c_k(\epsilon)$ depend analytically on the parameters ϵ . The analyticity of the function \dot{p} implies that the Fourier coefficients $c_k(\epsilon)$ are rapidly decreasing as $\|k\| \rightarrow \infty$. Formal termwise integration of (5.1) would lead to

$$p(t) = p(0) + t \sum_{k \in \mathbf{Z}^2, \langle \nu(\epsilon), k \rangle = 0} c_k(\epsilon) e^{2\pi i \langle x_0, k \rangle} \quad (5.2)$$

$$+ \sum_{k \in \mathbf{Z}^2, \langle \nu(\epsilon), k \rangle \neq 0} \frac{c_k(\epsilon)}{2\pi i \langle \nu(\epsilon), k \rangle} e^{2\pi i \langle x_0 + t\nu(\epsilon), k \rangle}. \quad (5.3)$$

The coefficient of the linear term in t , the *secular term*, contains, apart from the term $c_0(\epsilon)$ (which is equal to the average of the function \dot{p} over the torus), terms for nonzero k if and only if the components $\nu_1(\epsilon)$ and $\nu_2(\epsilon)$ have a rational ratio. If we assume that $\nu_2(\epsilon) \neq 0$ and $\nu_1(\epsilon)/\nu_2(\epsilon) = n_1/n_2$, with $n_1 \in \mathbf{Z}$, $n_2 \in \mathbf{Z}_{>0}$ and $\gcd(n_1, n_2) = 1$, then

$$t \langle \nu(\epsilon), k \rangle = t \frac{\nu_2(\epsilon)}{n_2} (n_1 k_1 + n_2 k_2),$$

which is equal to an integer if t is equal to an integral multiple of $T := n_2/\nu_2(\epsilon)$. This in turn implies that the function $t \mapsto \dot{p}(x_0 + t\nu(\epsilon), \epsilon)$ is periodic with period equal to T . For generic analytic functions $\nu_1(\epsilon)$, $\nu_2(\epsilon)$ the ratio $\nu_1(\epsilon)/\nu_2(\epsilon)$ is rational for a dense subset of parameter values, where the denominator n_2 and therefore also the period T is unbounded in every nonvoid open subset of the parameter space. The coefficients of the secular term in general would have a correspondingly wild behaviour as a function of the parameters.

In the case that the fraction $\nu_1(\epsilon)/\nu_2(\epsilon)$ is irrational, when the function

$$t \mapsto \dot{p}(x_0 + t\nu(\epsilon), \epsilon)$$

is not periodic, then the sum in (5.3) over the $k \in \mathbf{Z}^2$ such that $\langle \nu(\epsilon), k \rangle \neq 0$ need not converge, due to the possibility that the denominators $\langle \nu(\epsilon), k \rangle$ may become arbitrarily small. This problem can arise, despite the rapid decrease of the Fourier coefficients $c_k(\epsilon)$ when $\|k\| \rightarrow \infty$. If the fraction $\nu_1(\epsilon)/\nu_2(\epsilon)$ satisfies suitable diophantine inequalities, then the sum is convergent and defines a quasiperiodic function of t on a two-dimensional torus.

Proposition 5.1 implies that for the motion of Chaplygin's sphere when j is vertical *none of the above complications occur*: no wild behaviour of the coefficients of the secular term and no problem with convergence of the Fourier series in (5.3). Note that the conclusions of Proposition 5.1 have not been obtained by means of an analysis with Fourier series, but by using the reconstruction technique in Hermans [18, Sec. 3.2] instead.

5.3 Euler's Equations

In this subsection we give some explicit formulas, which among other show that if j is vertical, then the rotational motion is determined by Euler's equations. An extensive discussion of the Euler top can be found in Cushman and Bates [12, Ch. III].

If $j = j_3 e_3$, then it follows from (4.1) that

$$2T = \langle A\omega, j \rangle = j_3 \langle A\omega, e_3 \rangle = j_3 \langle \omega, u \rangle \quad (5.4)$$

Because (3.3) implies that $\langle \omega, \frac{du}{dt} \rangle = \langle \omega, u \times \omega \rangle = 0$, we obtain from (5.4) that

$$\left\langle \frac{d\omega}{dt}, u \right\rangle = \frac{d}{dt} \langle \omega, u \rangle = 0.$$

But then (3.4) implies that

$$(I + \rho)\omega \times \omega = I\omega \times \omega = \frac{d}{dt}(I + \rho)\omega, \quad (5.5)$$

which are Euler's equations with I replaced by $I + \rho$.

The equation $j_3 e_3 = j = A I_{\rho, u} \omega$ yields in view of (5.4) that

$$j_3 u = (I + \rho)\omega - \rho \langle \omega, u \rangle u = (I + \rho)\omega - \frac{2T\rho}{j_3} u,$$

which leads to the formula

$$u = \frac{j_3}{j_3^2 + 2T\rho} (I + \rho) \omega, \quad (5.6)$$

which expresses u in terms of ω . The differential equation $\frac{du}{dt} = u \times \omega$ then leads to the differential equation

$$\frac{du}{dt} = \frac{j_3^2 + 2T\rho}{j_3} u \times J u, \quad (5.7)$$

for u only, which also is an equation of Euler type. Here $J = (I + \rho)^{-1}$, cf. (2.18).

From (5.6) and (5.4) we obtain that

$$\langle u, J u \rangle = \frac{2T}{j_3^2 + 2T\rho}, \quad (5.8)$$

which shows that the solutions $u(t)$ run on the intersection of the sphere $\langle u, u \rangle = 1$ with an ellipsoid defined by the symmetric matrix J . Using (5.6) it follows that also $(I + \rho) \omega(t)$ runs over the intersection of a sphere with an ellipsoid, explicitly given by

$$\langle (I + \rho) \omega, (I + \rho) \omega \rangle = \left(\frac{j_3^2 + 2T\rho}{j_3} \right)^2, \quad (5.9)$$

$$\langle \omega, (I + \rho) \omega \rangle = 2T (1 + 2T\rho/j_3^2). \quad (5.10)$$

The kinetic energy T is given in terms of u by

$$2T = \frac{j_3^2 \langle u, J u \rangle}{1 - \rho \langle u, J u \rangle}, \quad (5.11)$$

cf. (5.8). The critical points of T on the unit u -sphere are the unit eigenvectors u of J , with the eigenvalues $1/(\iota + \rho)$, with $\iota = I_1, I_2$ or I_3 . The corresponding critical value is equal to

$$T_{\text{crit}} = \frac{j_3^2}{2\iota}, \quad (5.12)$$

which is in accordance with (4.9) when $j = j_3 e_3$. It is well-known that the eigenvectors for the two extremal eigenvalues of $I + \rho$ are stable equilibrium points of the Euler equation, whereas the eigenvectors for the intermediate eigenvalue are linearly unstable equilibrium points of Euler's equation (5.7). See for instance Cushman and Bates [12, p. 117]. The latter implies that, when j is vertical, the critical circles of T_j for the intermediate critical values are linearly unstable.

5.4 The Translational Motion

Let $u = u(t)$ be the solution of (5.7) on the intersection curve of the ellipsoid (5.8) with the unit sphere U . The rotational velocity vector $\omega = \omega(t)$ is determined in terms of u by means of (5.6). The projection $A \mapsto u = A^{-1} e_3$, cf. (2.12), exhibits the T_j -level set in $\text{SO}(3)$

as a principal $\text{SO}(2)$ -bundle over the aforementioned curve on U , where $\text{SO}(2)$ is the group of rotations about the vertical axis, acting on $\text{SO}(3)$ by means of left multiplications. The projection intertwines the flow on the T_j -level set in $\text{SO}(3)$ with the motion on the curve on U determined by (5.7), where the flow on the T_j -level set in $\text{SO}(3)$ is given by the differential equation (3.2) in which $\omega = \omega(t)$ is determined by (5.6).

If T_j is close to the intermediate critical level, then the solution $u = u(t)$ of (5.7) will stay for a long time near one of the unstable equilibria u_{crit} of (5.7) before it moves on to the other one. During this time $\omega = \omega(t)$ will stay close to the nonzero vector ω_{crit} which is determined by (5.6) with u replaced by u_{crit} . It follows that $A = A(t)$ will make many rotations during that time. This leads to the following conclusion.

Proposition 5.2 *Let \mathcal{T}_{per} denote the levels of T_j such that the motion on the corresponding torus in $\text{SO}(3)$ is periodic with the same period the motion in the (u, ω) -space, cf. Proposition 5.1. Then \mathcal{T}_{per} is an infinite subset of \mathbf{R} with the intermediate critical level $T = j_3^2/2I_2$ as its only accumulation point. This accumulation point is approached by \mathcal{T}_{per} both from above and from below.*

The translational motion, the motion of the point of contact p , is obtained by integrating the right hand side $r(A\omega) \times e_3$ of (3.1). Here the vector $\theta := A\omega$, the rotational velocity vector in space coordinates, the *herpolhode* in Poincot's description of the Euler top, lies in view of (5.4) in the fixed horizontal plane $\langle \theta, e_3 \rangle = 2T/j_3$. We will use the discussion in Cushman and Bates [12, II.7.2] of the construction of Poincot. (Additional information can be found in Routh [34, Art. 151, p.98 and pp. 471-473]. An interesting fact is for instance that the herpolhode is always concave towards the interior, without inflexion points. In his *Théorie nouvelle de la rotation des corps*, 1834, Poincot drew the herpolhode like a snake (= herpes in Greek), which therefore is misleading.)

To begin with, the image of the T_j -level set in $\text{SO}(3)$ under the projection $A \mapsto \theta = A\omega$ is invariant under the rotations around the vertical axis, and is therefore known if we know how $\langle \theta, \theta \rangle = \langle \omega, \omega \rangle$ varies as ω is coupled to u by means of (5.4) and $u \in U$ runs over the curve determined by (5.8).

We will restrict ourselves to one of the two opposite connected components of the intersection of U with the ellipsoid (5.8), which is the orbit of the motion on the u -sphere. On it, the function $\langle \theta, \theta \rangle$ has four critical points which all are nondegenerate. Two of these correspond to the maximal value R_{max}^2 and two to the minimal value R_{min}^2 , where $R_{\text{max}} > R_{\text{min}} > 2T/j_3$. To be more precise, if $T_{\text{crit},i} = j_3^2/2I_i$ denote the critical values of T_j , cf. (5.12), then we have the following two cases.

- i) $T_{\text{crit},3} < T < T_{\text{crit},2}$. We have $u_3 \neq 0$ on the intersection of U with the ellipsoid (5.8), and therefore u_3 has one sign on the u -orbit. The critical points corresponding to the maximal value are equal to the two intersection points $(\pm u_1, 0, u_3)$ of the orbit with the coordinate plane $u_2 = 0$. The critical points corresponding to the minimal value are equal to the two intersection points $(0, \pm u_2, u_3)$ of the orbit with the coordinate plane $u_1 = 0$.

- ii) $T_{\text{crit},2} < T < T_{\text{crit},1}$. We have $u_1 \neq 0$ on the intersection of U with the ellipsoid (5.8), and therefore u_1 has one sign on the u -orbit. The critical points corresponding to the maximal value are equal to the two intersection points $(u_1, 0, \pm u_3)$ of the orbit with the coordinate plane $u_2 = 0$. The critical points corresponding to the minimal value are equal to the two intersection points $(u_1, \pm u_2, 0)$ of the orbit with the coordinate plane $u_3 = 0$.

It follows that the image of the T_j -level set in $\text{SO}(3)$ under the projection $A \mapsto \dot{p} = r A \omega \times e_3$ is equal to a circular annulus in the plane, with center at the origin. When $u(t)$ runs around the orbit in U then it subsequently passes an intersection point with the coordinate plane $u_2 = 0$, then an intersection point with a second coordinate plane, then the other intersection point with the coordinate plane $u_2 = 0$, and finally the other intersection point with the second coordinate plane, before it closes. The corresponding point \dot{p} in the annulus will then reach the outer circle, with a second order contact, then touch the inner circle, return to the outer circle at a point which is rotated over an angle α as compared to the first contact point with the outer circle, and then touch the inner circle for the second time before the curve in U closes.

If the rotational motion $A(t)$ is periodic with the same period as the motion $u(t)$ on U , then the third point of contact with the outer circle is equal to the first one, which means that 2α is equal to an integral multiple of 2π . There are two cases.

- a) α itself is not an integral multiple of 2π , which correspond to the case that the second point of contact with the outer circle lies opposite to the first one. In this case the \dot{p} -orbit is symmetric about the origin, its time average is equal to zero and the motion of the point of contact $p(t)$ with the horizontal plane is periodic. In other words, the speed of the straight line motion in Proposition 5.1 is equal to zero.
- b) α is equal to an integral multiple of 2π , which correspond to the case that the second point of contact with the outer circle is equal to the first one. Equivalently, $\dot{q}(t)$ is periodic with a period equal to half the period of the motion $u(t)$ on U . In this case the \dot{p} -orbit is not symmetric about the origin.

The argument preceding Proposition 5.2 yields that α is given by a smooth real-valued function of T which tends to $\pm\infty$ as $T \rightarrow T_{\text{crit},2}$.

Moreover, if T converges to the intermediate critical level $T = j_3^2/2I_2$, then the inner boundary circle of the \dot{p} -annulus shrinks to the origin. Because $u(t)$ stays for a long time near the critical point $\pm e_2$, to which the intersection point of the u -orbit with the coordinate plane $u_1 = 0$ in case i) and $u_3 = 0$ in case ii) is close, the conclusion is that $\dot{p}(t)$ stays for a long time close to the inner boundary circle, running many times around it in the process. It follows that the time average of $\dot{p}(t)$, which is equal to the speed of the straight line motion in Proposition 5.1, converges to zero when T converges to the intermediate critical level $T = j_3^2/2I_2$.

We therefore arrive at the following conclusions.

Proposition 5.3 *Let $\mathcal{T}'_{\text{per}}$ denote the levels of T_j such that the motion of*

$$\dot{q}(t) = r A(t) \omega(t) \times e_3$$

is periodic with half the period of the motion in the (u, ω) -space. Then $\mathcal{T}'_{\text{per}}$ is an infinite subset of \mathbf{R} with the intermediate critical level $T = j_3^2/2I_2$ as its only accumulation point. This accumulation point is approached by $\mathcal{T}'_{\text{per}}$ both from above and from below. If the speed of the straight line motion in Proposition 5.1 is nonzero, then necessarily $T \in \mathcal{T}'_{\text{per}}$. If $T \in \mathcal{T}'_{\text{per}}$ converges to the intermediate critical level $T = j_3^2/2I_2$, then the speed of the corresponding straight line motion, cf. Proposition 5.1, converges to zero.

We conjecture that for most values of $T \in \mathcal{T}'_{\text{per}}$ the speed of the straight line motion in Proposition 5.1 is *not* equal to zero. Here the word “most” can mean all except finitely many, or for generic values of I_1, I_2, I_3 .

If $T \notin \mathcal{T}'_{\text{per}}$ approaches a value $T_0 \in \mathcal{T}'_{\text{per}}$ for which speed of the straight line motion in Proposition 5.1 is not equal to zero, then the bounded area in which the quasiperiodic motion of $p(t)$ takes place “opens up to infinity”, and closes again to a bounded subset when T has passed the value T_0 . If the above conjecture holds, then this scenario takes place infinitely often when T approaches the intermediate critical level $T = j_3^2/2I_2$, but with the average speed of the point of contact converging to zero.

5.5 Chaplygin

The remainder of Chaplygin [9, §6], starting with the sentence “In addition, there is an exceptional case, \dots ”, consists of a discussion of the case that j is vertical. This discussion contains several interesting observations, but it does not give the qualitative information about the motion as in Proposition 5.1.

6 Bobylev’s Sphere

A rigid body is called a *solid of revolution* if it is dynamically symmetric with respect to all rotations R about a given axis, for which we can take the vertical axis. This means that both the surface S and the inertial tensor I are invariant under such rotations R . Because for Chaplygin’s sphere the surface S already is invariant, it is a solid of revolution if and only if $I_1 = I_2$. In this case Chaplygin’s sphere is equal to Routh’s sphere with the center of mass at the center of the sphere. This case has been studied by Bobylev [5], where in our case there is no gyroscope as mentioned in the title of [5].

The assumption of having a solid of revolution implies that the equations of motion are invariant under the *right* $\text{SO}(2)$ -action

$$(A, \omega, a) \mapsto (A R^{-1}, R \omega, R a)$$

of rotations R about the vertical axis. The right $\text{SO}(2)$ -action also leaves the kinetic energy T and the moment j invariant. The quotients of the connected components of the regular

(j, T) -level surfaces by the right $\text{SO}(2)$ -action are diffeomorphic to circles, which implies that the solutions of the right $\text{SO}(2)$ -reduced system, for the regular levels of (j, T) , are periodic. The reconstruction method in [18, Sec. 3.2] then leads to the following conclusions. Note that the vector $A\omega$ is invariant under the right $\text{SO}(2)$ -action, because $(A R^{-1})(R\omega) = A\omega$.

Proposition 6.1 *Suppose that two of the moments of inertia are equal to each other. Then the right $\text{SO}(2)$ -reduced rotational motion on the regular levels of (j, T) is periodic, with a period which depends analytically on j and T . In particular the vector $A\omega$, and therefore also $\frac{dp}{dt} = r(A\omega) \times e_3$ performs a periodic motion.*

It follows that the rotational motion on the regular levels of (j, T) is quasiperiodic on two-dimensional analytic tori, depending analytically on j and T . Also, the motion of the point of contact p is equal to the superposition of a straight line motion with constant speed and a periodic motion with the same period as that of the right $\text{SO}(2)$ -reduced motion.

As in the case when j is vertical, cf. Proposition 5.1, there are no problems with secular terms or with the convergence of Fourier series for the motion of the point of contact p . In contrast with Proposition 5.1, we obtain here that $\frac{dp}{dt}$ is always periodic.

It follows from (3.18) that

$$F = I_1 x + I_3 y z, \quad x = u_1 \omega_1 + u_2 \omega_2, \quad y = \omega_3, \quad z = u_3.$$

This is known as *Jellet's integral*, cf. Routh [34, Art. 243].

In the same notation, the constant of motion G of (3.19) takes the form

$$G = (I_1 + \rho)^2 w + (I_3 + \rho)^2 y^2 - 2\rho(x + yz)((I_1 + \rho)x + (I_3 + \rho)yz) + \rho^2(x + yz)^2.$$

Together with the kinetic energy we thus obtain three constants of motion in the four-dimensional $\text{E}(2) \times \text{SO}(2)$ -reduced phase space, the “fully reduced” phase space. The regular level sets of all the constants of motion are algebraic curves, the periodic motion on which can be obtained by means of quadratures. We do not go into further details about this here.

6.1 Chaplygin

In his Introduction, Chaplygin [9] referred to the papers of Bobylev [5] and Zhukovsky [37] for the case that two moments of inertia are equal. In the beginning of [9, §6], Chaplygin wrote “We will not treat the case when two or all three principal moments of inertia are equal, because the motion of such a sphere has already been investigated (see the Introduction).” Apparently Chaplygin did not feel that the articles of Bobylev and Zhukovsky, which I have not seen, needed further comments. In particular I wonder whether Bobylev and/or Zhukovsky used the moment j of the momentum around the point of contact as a constant of motion. In the paper [8], to which Chaplygin referred for the fact that j is a constant of motion, there is no reference to Bobylev or Zhukovsky.

The description of Bobylev's paper [5] in the Fortschritte der Mathematik says: “After the proposed integration, which can be performed with the help of the elliptic functions of

Weierstrass, the author reaches the conclusion that the center of the sphere describes a curve which is enclosed between two parallel straight lines and has a periodic character, where it successively reaches the one and the other straight line with constant distances between the successive contact points on each of the straight lines.” This corresponds to the description of the point of contact p in Proposition 6.1.

As observed before, Chaplygin’s sphere with two equal moments of inertia is equal to Routh’s sphere with the center of mass at the center of the sphere. In [34, Art. 243], where Routh’s sphere is treated, no special attention is paid to the case that the center of mass is at the center of the sphere.

7 An Invariant Volume Form

We return to the general case of Chaplygin’s sphere, with arbitrary moments of inertia, arbitrary total mass and radius of the sphere and arbitrary moment of the momentum around the point of contact. In the following lemma we use the notation of Subsection 3.3. Note that $X(u)$ in (3.15) is strictly positive when $\|u\| = 1$, because the eigenvalues of the symmetric matrix $J = (I + \rho)^{-1}$ are equal to $1/(I_i + \rho)$, $i = 1, 2, 3$, and therefore strictly smaller than $1/\rho$.

Lemma 7.1 *Consider the equations of motion $\frac{du}{dt} = u \times \omega$, $\frac{dv}{dt} = v \times \omega$ in the (u, v) -space $\mathbf{R}^3 \times \mathbf{R}^3$, in which ω is determined in terms of u and v by (3.11). Let Ω be the volume form in \mathbf{R}^6 which is equal to $X(u)^{-1/2}$ times the Euclidean volume form, in which $X(u)$ is defined by (3.15). Then Ω is invariant under the flow in \mathbf{R}^6 .*

Proof The velocity field of the flow is equal to the vector field

$$R_\omega : (u, v) \mapsto (u \times \omega, v \times \omega),$$

in which $\omega = \omega(u, v)$ is determined by (3.11). The divergence of this vector field is equal to the trace of the derivative, and therefore equal to the sum of the trace of the derivative D_u of $u \mapsto u \times \omega(u, v)$ and the trace of the derivative D_v of $v \mapsto v \times \omega(u, v)$. Because the traces of the linear mappings $\delta u \mapsto \delta u \times \omega(u, v)$ and $\delta v \mapsto \delta v \times \omega(u, v)$ are equal to zero, the divergence is equal to trace $D_u^\omega + \text{trace } D_v^\omega$, in which $D_u^\omega : \delta u \mapsto u \times \frac{\partial \omega(u, v)}{\partial u} \delta u$ and $D_v^\omega : \delta v \mapsto v \times \frac{\partial \omega(u, v)}{\partial v} \delta v$. From (3.11) we obtain that

$$\omega(u, v) = Jv + \psi(u, v)Ju, \tag{7.1}$$

in which $\psi(u, v) := Y(u, v)/X(u)$ and $Y(u, v)$, $X(u)$ are given by (3.16), (3.15), respectively.

We now use that the trace of $\delta u \mapsto u \times J\delta u$ is equal to

$$\sum_{i=1}^3 \langle u \times J(e_i), e_i \rangle = \sum_{i=1}^3 (I_i + \rho)^{-1} \langle u \times e_i, e_i \rangle = 0.$$

It follows that $\text{trace } D_u^\omega$ is equal to the trace of the rank one mapping

$$\delta u \mapsto \left(\frac{\partial \psi(u, v)}{\partial u} \delta u \right) u \times J u,$$

and therefore equal to

$$\frac{\partial \psi(u, v)}{\partial u} (u \times J u) = \langle u \times J u, J v \rangle / X(u), \quad (7.2)$$

because

$$\frac{\partial X(u)}{\partial u} (u \times J u) = -2 \langle u \times J u, J u \rangle = 0.$$

Similarly the trace of $\delta v \mapsto v \times J \delta v$ is equal to zero, and therefore the trace of D_v^ω is equal to the trace of the rank one mapping

$$\delta v \mapsto \left(\frac{\partial \psi(u, v)}{\partial v} \delta v \right) u \times J u,$$

which is equal to zero because

$$\langle u, J (u \times J u) \rangle = \langle J u, u \times J u \rangle = 0.$$

The conclusion is therefore that the divergence of the vector field is equal to (7.2). On the other hand the derivative $R_\omega X$ of the function X in the direction of the vector field R_ω is equal to

$$-2 \langle u \times \omega, J u \rangle = -2 \langle u \times J v, J u \rangle = 2 \langle u \times J u, J v \rangle,$$

and therefore

$$\text{div } R_\omega = \frac{1}{2} X^{-1} R_\omega X.$$

It follows that

$$\text{div } (X^{-1/2} R_\omega) = -\frac{1}{2} X^{-3/2} R_\omega X + X^{-1/2} \text{div } R_\omega = 0,$$

which completes the proof of the lemma. \square

Let M be a smooth manifold of dimension m , Ω a smooth volume form on M and f a smooth function on M such that $df \neq 0$ at every point of the level set $M_c := \{x \in M \mid f(x) = c\}$. Then M_c is a smooth $(m-1)$ -dimensional submanifold of M , and there is a unique volume form ω on M_c such that

$$\Omega_x(v_1, \dots, v_{m-1}, v_m) = \omega_x(v_1, \dots, v_{m-1}) df_x(v_m) \quad (7.3)$$

whenever $x \in M_c$, $v_i \in T_x M_c$ every $1 \leq i \leq m-1$, and $v_m \in T_x M$. The volume form ω on M_c is smooth and nonzero at every $x \in M$ where $\Omega_x \neq 0$. It is called the *relative quotient of Ω and df* and denoted by $\omega = \Omega/df$.

Jacobi observed in [21, 10–14. Vorlesung], that if v is a smooth vector field on M such that $v f = 0$ and its divergence $\mathcal{L}_v \Omega / \Omega$ with respect to Ω is equal to zero, then the flow of v leaves M_c , df and Ω invariant, and therefore ω as well. In other words, if v_c denotes the restriction of v to M_c , which is tangent to M_c , then the divergence $\mathcal{L}_{v_c} \omega / \omega$ of v_c with respect to ω is equal to zero.

Applying this principle successively to the functions in the left hand sides in (3.9), which are all invariant under the action $(u, v) \mapsto (Ru, Rv)$ of arbitrary rotations R , and using that the set determined by (3.9) can be identified with $\text{SO}(3)$ if j is not vertical, we arrive at the following corollary, where for vertical j we can apply a continuity argument.

Corollary 7.2 *Let dA be a Haar volume form on $\text{SO}(3)$, a volume form on $\text{SO}(3)$ which is invariant under right (or left) multiplications with elements of $\text{SO}(3)$. Then $X^{-1/2} dA$ is invariant under the R_{ω_j} -flow on $\text{SO}(3)$.*

On the regular level surfaces for the kinetic energy function T_j , the area form

$$X^{-1/2} dA / dT_j$$

is invariant under the R_{ω_j} -flow on $\text{SO}(3)$.

If M is a two-dimensional smooth manifold, v a nowhere vanishing smooth vector field on M , and α is a nowhere vanishing smooth area form on M which is v -invariant, then the fact that the three-form $d\alpha$ is equal to zero on M implies that

$$0 = \mathcal{L}_v \alpha = d(i_v \alpha) + i_v(d\alpha) = d(i_v \alpha), \quad (7.4)$$

or that the nowhere vanishing one-form

$$\beta := i_v \alpha$$

is closed. Let $g(x)$ be the function which is obtained by integrating β along a curve in M starting at some base point and ending up at x . (This is an allowed procedure in the “integration by quadratures” philosophy.) Then $dg = \beta$, hence $v g = 0$ and the orbits of the v -solution curves correspond to the level sets of g . Note that $dg = \beta$ is nowhere vanishing, which implies that the connected components of the level sets are smooth curves. Also note that the function g is always globally defined on the universal covering space of M , but that on M in general it will be a multi-valued function, with the indeterminacy that $g(x)$ has to be replaced by $g(x) + \langle [\gamma], [\omega] \rangle$ if the curve ending up at x is followed by a loop γ which starts and ends at x . Here $[\gamma] \in H_1(M)$ and $[\omega] \in H^1(M)$ denote the homology and (de Rham) cohomology class of γ and ω , respectively.

If Ω_0 is a nonzero smooth volume form on a manifold M , then every smooth volume form Ω on M is of the form $\Omega = \mu \Omega_0$, for a unique smooth function μ . Therefore the search for an invariant volume form is a matter of finding the right factor (multiplier) μ . If one has such a multiplier on an n -dimensional manifold and one also has $n - 2$ independent constants of motion f_1, \dots, f_{n-2} , then taking successively the relative quotient volume forms

on the level manifolds $f_i = c_i$, one obtains multipliers on the $(n - j)$ -dimensional level sets $f_1 = c_1, \dots, f_j = c_j$. For $j = n - 2$ one finally obtains the “last multiplier” on the two-dimensional level surface of all the constants of motion, to which one then can apply the above integration by quadratures. This is method for integration of vector fields by quadratures, which has been introduced in [21, 10-14. Vorlesung], is called *Jacobi’s last multiplier method*.

If the two-dimensional M is compact and connected then the fact that v has no zeros implies that M is diffeomorphic to a torus, as we have observed before at the end of Subsection 4.1. Siegel [36, Lemma 3 and 4] proved that there exists a smooth closed loop C in M such that V is everywhere transversal to C and that for every such C and every v -solution curve γ there exists a $t > 0$ such that $\gamma(t) \in C$. The transversality of v to C implies that if $x \in C$, and $t \mapsto \gamma(t, x)$ denotes the v -solution curve with $\gamma(0, x) = x$, and $T(x)$ is the smallest $t > 0$ such that $\gamma(t, x) \in C$, then T depends smoothly on $x \in C$ and we obtain a smooth Poincaré map $P : x \mapsto \gamma(T(x), x) : C \rightarrow C$. The restriction β_C of the above one-form $\beta = i_v \alpha$ to C is a smooth one-form on C without zeros, and there an angle coordinate θ on C such that $d\theta = \beta_C$, which is unique up to an additive constant. The v -invariance of the area form α implies that β is v -invariant. In turn this implies that β_C is invariant under P and we conclude that $P(\theta) = \theta + c$, where c is a constant. In other words, the return map is a rotation.

By modifying the speed of the solution curves before they arrive at C , we can arrange that the return time $T(x)$ is a constant. In other words, there exists a strictly positive smooth function f on M (which we can choose to be non-constant only in a thin strip at one side of C), such that $T(x)$ is equal to a constant if we replace v by $f v$. Let w_C be the unique tangent vector field of C such that $i_{w_C} \alpha_C \equiv 1$. Because w_C is invariant under P , we can carry w_C around with the v -flow and obtain an extension w of w_C which is a smooth vector field on M and commutes with $f v$ by construction. It is also clear that w and $f v$ are everywhere linearly independent.

We now recall the argument of Arnol’d and Avez [3, Appendix 26] that the $f v$ -flow is quasiperiodic. It follows that if $e^{t v}$ denotes the flow after time t of the vector field v , then

$$(t, s) \mapsto e^{t f v} \circ e^{s w} \quad (7.5)$$

defines an action of \mathbf{R}^2 on M . Because of the linearly independence of $f v$ and w , the orbits are open subsets of M . because the orbits form a partition of M and M is connected, there is only one orbit, equal to M . In other words, the action is transitive. If $e^{t f v} \circ e^{s w}(x) = x$ for some $x \in M$ then $e^{t f v} \circ e^{s w}(x) = x$ for every $x \in M$. The *period lattice*

$$\Pi := \{(t, s) \in \mathbf{R}^2 \mid e^{t f v} \circ e^{s w} = 1\}$$

is a discrete additive subgroup of \mathbf{R}^2 , and because for each $x \in M$ the mapping $(t, s) \mapsto e^{t f v} \circ e^{s w}(x)$ induces a diffeomorphism from \mathbf{R}^2/Π onto the compact manifold M , the conclusion is that the lattice Π is two-dimensional. In the coordinates with respect to a \mathbf{Z} -basis of Π , \mathbf{R}^2/Π is equal the standard torus $\mathbf{R}^2/\mathbf{Z}^2$. Because in these coordinates the vector field $f v$ is constant, we arrive at the conclusion that the $f v$ -flow is quasiperiodic on a two-dimensional torus.

It is clear from its introduction that the function f is far from unique. Actually, Kolmogorov [23] proved that if the rotation number of the Poincaré map P satisfies suitable diophantine inequalities, ensuring that it cannot be approximated too rapidly by means of rational numbers, then also the v -flow, without the time-reparametrizing factor f , is quasiperiodic. He also showed that for this conclusion the diophantine inequalities for the rotation number are essential, in the sense that in general it is not sufficient to assume that the rotation number of P is irrational.

In Corollary 8.4 we will obtain that for the Chaplygin sphere the rotational motion is quasiperiodic on two-dimensional tori, if the reparametrization of the time corresponds to multiplication of the vector field by the specific function $f = X(u)^{1/2}$, where $X(u)$ is given by (3.15).

7.1 Chaplygin

The last part of Chaplygin [9, §2], starting with “To solve the problem completely, ...”, contains the proof of Lemma 7.1, followed by the conclusion, in one line, that Jacobi’s last multiplier method can be applied in order to solve the equations of motion by quadratures. Apparently at the time of [9] this method was so well-known, that no further explanations or references were needed.

8 Two Commuting Vector Fields

8.1 The Second Vector Field

It follows from (4.4) that the tangent spaces of the level surfaces in $\text{SO}(3)$ of the function T_j are spanned by the vector fields R_{ω_j} and $R_{(I+\rho)\omega_j}$. It is therefore natural to investigate the divergence of the vector field $R_{(I+\rho)\omega_j}$ with respect to the area form $X^{-1/2} dA/dT_j$, in analogy with Corollary 7.2. Note that $I + \rho = J^{-1}$, cf. (2.18).

Lemma 8.1 *Consider the vector field R_ν in the (u, v) -space $\mathbf{R}^3 \times \mathbf{R}^3$, defined by $R_\nu u := u \times \nu$, $R_\nu v := v \times \nu$, where $\nu = \nu(u, v) := (I + \rho)\omega(u, v)$ and $\omega = \omega(u, v)$ is determined in terms of u and v by (3.11). Let $X(u)$ be defined by (3.15) and let Ω be the volume form in \mathbf{R}^6 which is equal to $X(u)^{-1/2}$ times the Euclidean volume form. Then the divergence of R_ν with respect to Ω is equal to zero.*

It follows that if dA is a Haar volume form on $\text{SO}(3)$, then the divergence of the vector field $R_{(I+\rho)\omega_j}$ on $\text{SO}(3)$ with respect to $X^{-1/2} dA$ is equal to zero. Also, the divergence is equal to zero of the vector field $R_{(I+\rho)\omega_j}$ on any regular level surfaces of T_j , with respect to the area form $X^{-1/2} dA/dT_j$.

Proof The proof follows the same lines as the proof of Lemma 7.1, The calculations are actually somewhat easier this time, because the expression

$$\nu(u, v) = (I + \rho)\omega(u, v) = v + \psi(u, v)u$$

for ν is simpler than the formula (7.1) for ω .

The divergence of R_ν is equal to $\text{trace } D_u^\nu + \text{trace } D_v^\nu$, in which D_u^ν and D_v^ν is equal to the trace of the derivative of $u \times \nu(u, v) = u \times v$ and $v \times \nu(u, v) = \psi(u, v) \cdot v \times u$ with respect to u and v , respectively. It follows that the divergence is equal to the trace of the rank one linear mapping

$$\delta v \mapsto \left(\frac{\partial \psi(u, v)}{\partial v} \delta v \right) v \times u,$$

and therefore equal to

$$\frac{\partial \psi(u, v)}{\partial v} (v \times u) = \langle u, J(v \times u) \rangle / X(u).$$

On the other hand

$$R_\nu X = -2 \langle u \times \nu, J u \rangle = -2 \langle u \times v, J u \rangle = 2X \text{ div } R_\nu,$$

which in the same way as at the end of the proof of Lemma 7.1 implies that

$$\text{div } (X^{-1/2} R_\nu) = -\frac{1}{2} X^{-3/2} R_\nu X + X^{-1/2} \text{ div } R_\nu = 0,$$

or that the divergence of R_ν with respect to Ω is equal to zero.

The statements about the volume form on $\text{SO}(3)$ and the area form on the level sets of T_j follow in the same way as Corollary 7.2 follows from Lemma 7.1. \square

Lemma 8.2 *Let dA be the Euclidean volume form on $\text{SO}(3)$. Then*

$$(dA / dT_j) (R_{\omega_j}, R_{(I+\rho)\omega_j}) = 1.$$

Proof It follows from 4.4 that if $\nu = \omega_j \times (I + \rho) \omega_j$, then

$$R_\nu T_j = \langle \omega_j \times (I + \rho) \omega_j, \omega_j \times (I + \rho) \omega_j \rangle = dA(\omega_j, (I + \rho) \omega_j, \omega_j \times (I + \rho) \omega_j),$$

which implies the statement of the lemma in view of the defining equation (7.3) of the relative quotient of a volume form and the total derivative of a function. \square

Proposition 8.3 *Let the factor $X(u)$ be defined by (3.15). Define the vector fields ξ and η on $\text{SO}(3)$ by $\xi := X(u)^{1/2} R_{\omega_j}$ and $\eta := X(u)^{1/2} R_{(I+\rho)\omega_j}$, respectively. Then the vector fields ξ and η commute.*

Define the area form α_j and the one-forms β and γ on the regular level surfaces of T_j by $\alpha_j := X(u)^{-1} dA / dT_j$, $\beta := i_\xi \alpha_j$ and $\gamma := i_\eta \alpha_j$, where dA is the Euclidean volume form on $\text{SO}(3)$. Then $\alpha_j(\xi, \eta) = 1$, $\alpha_j = \beta \wedge \gamma$ and the one-forms β and γ are closed. The area form α_j and the one-forms β and γ are invariant under the flow of both vector fields ξ and η .

Proof It follows from (7.4) and the fact that the divergence of v with respect to α is equal to zero if and only if the one-form $i_v \alpha$ is closed. Because for any functions f on M we have that $i_{f v} g \alpha = f g i_v \alpha$, it follows that for every nowhere vanishing smooth function f we have that the divergence of v with respect to α is equal to zero if and only if the one-form $i_{f v} (f^{-1} \alpha)$ is closed.

If we apply this with $v = R_{\omega_j}$,

$$\alpha = X^{-1/2} dA/dT_j = X^{1/2} \alpha_j,$$

and $f = X^{1/2}$, then it follows from Corollary 7.2 that $\beta = i_\xi \alpha_j$ is closed, or equivalently that $\mathcal{L}_\xi \alpha_j = 0$. In a similar manner it follows from Lemma 8.1 that $\gamma = i_\eta \alpha_j$ is closed, or equivalently $\mathcal{L}_\eta \alpha_j = 0$.

On the other hand Lemma 8.2 implies that $\alpha_j(\xi, \eta) = 1$, from which it follows in turn that

$$0 = \mathcal{L}_\xi \alpha_j(\xi, \eta) = \alpha_j(\xi, [\xi, \eta]), \quad (8.1)$$

$$0 = \mathcal{L}_\eta \alpha_j(\xi, \eta) = \alpha_j([\eta, \xi], \eta) = \alpha_j(\eta, [\xi, \eta]). \quad (8.2)$$

Here we have used in (8.1) that $\mathcal{L}_\xi \alpha_j = 0$, $\mathcal{L}_\xi \xi = [\xi, \xi] = 0$, and $\mathcal{L}_\xi \eta = [\xi, \eta] = 0$, whereas in (8.2) we have used that $\mathcal{L}_\eta \alpha_j = 0$, $\mathcal{L}_\eta \xi = [\eta, \xi] = -[\xi, \eta]$, $\mathcal{L}_\eta \eta = [\eta, \eta] = 0$, and the antisymmetry of α_j . It follows from (8.1) that $[\xi, \eta]$ is a multiple of ξ and from (8.2) that $[\xi, \eta]$ is a multiple of η . Because ξ and η are everywhere linearly independent on the regular level sets of T_j , it follows that $[\xi, \eta] = 0$ there. Because the regular level sets are dense, it follows by continuity that the vector fields ξ and η commute on all of $SO(3)$. \square

Corollary 8.4 *If the rotational motion is parametrized by a time variable τ which is related to the time t by $\frac{d\tau}{dt} = X(u(t))^{-1/2}$, then the rotational motion on the regular level sets is quasi-periodic on two-dimensional analytic tori, depending analytically on the parameters j and T .*

Proof We have that

$$\frac{dA}{d\tau} = X^{1/2} \frac{dA}{dt} = X^{1/2} R_{\omega_j} A = \xi A.$$

The conclusion of the corollary follows from the discussion of the action (7.5) with M , v and w replaced by a regular level surface, ξ and η , respectively. \square

Question 8.5 Is there a proof of Corollary 7.2, Lemma 8.1 and Lemma 8.2, and therefore also of Proposition 8.3, which is based on a general principle, in the same way as Proposition 2.1 follows from Noether's principle for nonholonomic systems in Lemma 1.1? \odot

8.2 A Zero Average

According to (3.1), the time derivative of the j -component $\langle p, j \rangle$ of the point of contact p of the sphere with the horizontal plane is equal to r times the quantity We begin with

$$\begin{aligned} \langle (A\omega) \times e_3, j \rangle &= \langle A\omega, e_3 \times j \rangle = \langle \omega, (A^{-1}e_3) \times A^{-1}j \rangle \\ &= \langle \omega, u \times v \rangle = \langle \omega, u \times I\omega \rangle = -\det(u, \omega, I\omega). \end{aligned}$$

Here we have used (2.12) and (3.8) in the third equation, and in the fourth equation we have used (3.11) and (2.16), together with the facts that $u \times u = 0$ and ω is orthogonal to $u \times \omega$. We therefore obtain in view of (3.1) that

$$\frac{d}{d\tau} \langle p(\tau), j \rangle = -r X(u)^{1/2} \det(u, \omega, I\omega). \quad (8.3)$$

Lemma 8.6 *Let M be a connected component of a regular level set of T_j in $SO(3)$. Let $\det(u, \omega_j, I\omega_j)$ be viewed as a function on M . Then, for any continuous function f on U , the integral of $f(u) \det(u, \omega_j, I\omega_j)$ over M with respect to the area form α_j , cf. Proposition 8.3, is equal to zero.*

Proof We have

$$\begin{aligned} \xi u \times \eta u &= X(u) (u \times \omega_j) \times (u \times (I + \rho)\omega_j) \\ &= -X(u) \langle \omega_j, u \times (I + \rho)\omega_j \rangle u = X(u) \det(u, \omega_j, I\omega_j) u, \end{aligned}$$

because $\langle u, u \times (I + \rho)\omega \rangle = 0$ and therefore the ω_j -term drops out. Because $\alpha_j(\xi, \eta) = 1$, it follows that α_j is equal to the pull-back of $1/X(u) \det(u, \omega_j, I\omega_j)$ times the standard area form $d_2 u$ on U by means of the projection $\pi : A \mapsto u = A^{-1}e_3$ from the level set M to U . Now we have, for any smooth mapping π from a compact oriented manifold M to a compact oriented manifold U , and any volume form Ω on U , the formula

$$\int_M \pi^* \Omega = \deg(\pi) \cdot \int_U \Omega, \quad (8.4)$$

see for instance Guillemin and Pollack [16, p. 188]. In our case

$$f(u) \det(u, \omega_j, I\omega_j) \alpha_j = \pi^* \left(\frac{f(u)}{X(u)} d_2 u \right).$$

Moreover, the degree of π is equal to zero, because $\pi(M) \neq U$, see for instance the description at the end of Remark 11.2. Therefore the conclusion of the lemma is obtained by applying (8.4) to $\Omega = \frac{f(u)}{X(u)} d_2 u$. \square

It follows from Lemma 8.6 that the average of the right and side of (8.3) over the level surface, with respect to the area form α_j , is equal to zero. In view of Corollary 8.4 we can apply (5.3) with t replaced by τ and $p(t)$ replaced by $\langle p(\tau), j \rangle$, in the case that the rotational motion is not periodic. In this case the coefficient of the secular term is equal to $c_0(\epsilon)$, which is equal to the average of the right hand side of (8.3) over the level set, with respect to the area form α_j . This leads to the following conclusion.

Corollary 8.7 *Assume that the rotational motion on the regular level set is nonperiodic and that the series in (5.3), with t replaced by τ and $p(t)$ replaced by $\langle p(\tau), j \rangle$, is uniformly convergent. Then the function $\tau \mapsto \langle p(\tau), j \rangle$ is quasiperiodic on a two-dimensional torus. In particular, $\langle p(\tau), j \rangle$ remains bounded in this case.*

Note that the series mentioned in Corollary 8.7 converges uniformly when the irrational ratio $\nu_1(\epsilon)/\nu_2(\epsilon)$ mentioned after (5.3) is sufficiently slowly approximated by rational numbers. The set of irrational numbers for which this happens has full Lebesgue measure on the real axis.

Question 8.8 Do the complications with secular terms when the rotational motion is periodic, and convergence of Fourier series for nonperiodic rotational motions, as discussed after (5.3), really occur? \oslash

For the critical circles we have the simplification that $\frac{dp}{dt}$ is constant, cf. (4.10). When j is vertical, and when two of the moments of inertia are equal, we can reconstruct the full motion by means of Lie group techniques from a periodic motion, which implies that the aforementioned complications do not arise. See Proposition 5.1 and Proposition 6.1, respectively.

In the general case I have neither been able to find any analogous special features which would eliminate the complications with the primitives of the Fourier series, nor did I find a proof that the complications really occur. See also Question 11.12.

8.3 Chaplygin

The themes of Section 8 do not occur in Chaplygin [9]. The only exception may be that the existence of two closed one-forms β and γ such that $i_\xi \beta = 0$ and $i_\xi \gamma = 1$, cf. Proposition 8.3, is implicitly contained in the formulas in Chaplygin [9, §3, (29)], cf. Remark 11.3. As discussed after Remark 11.3 below, the vector field ξ corresponds to an explicitly determined constant vector field on the Jacobi variety $J(C)$ of a hyperelliptic curve C . In Remark (11.4) we determine the constant vector field on $J(C)$ to which η corresponds.

9 Some Simplifications

9.1 A Polynomial System

The square root in the factor $X(u)^{1/2}$ in front of the commuting vector fields ξ and η in Proposition 8.3 becomes double-valued if we extend the vector field holomorphically into the complex domain. One can make it single-valued by passing to the space of $(z, u, v) \in \mathbf{C} \times \mathbf{C}^3 \times \mathbf{C}^3$ where (u, v) still satisfy (3.9) and (3.13) and the new variable z is coupled to u by means of the equation

$$z^2 = X(u) = \rho^{-1} - \langle u, Ju \rangle. \quad (9.1)$$

This means that we pass to the branched (=ramified) double covering of the complexification of the level surface, which branches along the complex one-dimensional submanifold (complex curve) defined by the equation $X(u) = 0$. As observed in front of Lemma 7.1, we have $X(u) > 0$ when u is real, which implies that the curve defined by the equation $X(u) = 0$ has no real points.

We have that $\xi = R_{z\omega}$, $\eta = R_{z(I+\rho)\omega}$, where

$$z\omega = zJv + z^{-1}Y(u, v)Ju, \quad \text{and} \quad (9.2)$$

$$z(I+\rho)\omega = zv + z^{-1}Y(u, v)u, \quad (9.3)$$

in which we have used the abbreviation (3.16). We recall that $R_\nu u = u \times \nu$ and $R_\nu v = v \times \nu$. The actions on z are given by $2z\xi z = \xi z^2 = -2z\langle u \times Jv, Ju \rangle$ and $\eta z = -\langle u \times v, Ju \rangle$, because $\langle u \times Ju, Ju \rangle = 0$ and $\langle u \times u, Ju \rangle = 0$. It follows that the vector fields ξ and η are rational. The functions ξz and ηz are regular, but ξu and ξv seem to have poles along $z = 0$.

Combining the kinetic energy equation in the form (3.14) with (9.1), we see that $Y(u, v) = \pm zZ(v)^{1/2}$, and it follows that $z\omega = zJv \pm Z(v)^{1/2}Ju$ and $z(I+\rho)\omega = zv \pm Z(v)^{1/2}u$. Note that the sign choice in $\pm Z(v)^{1/2}$ is coupled to the choice of the sign of z by means of (3.16).

However, the vector fields are still double-valued at $z = 0$, where also the manifold of the solutions $((u, z), v)$ of the equations (3.9), (3.14) and (9.1) is singular. These singularities can be resolved by introducing one more variable ζ which is coupled to v by means of the equation

$$\zeta^2 = Z(v) = 2T - \langle v, Jv \rangle. \quad (9.4)$$

The kinetic energy equation $Y(u, v)^2 = X(u)Z(v) = z^2\zeta^2$ leads to $Y(u, v) = \pm z\zeta$. With the choice of the *minus* sign,

$$Y(u, v) = -z\zeta, \quad (9.5)$$

the vector fields $\xi = R_{z\omega}$ and $\eta = R_{z(I+\rho)\omega}$ are given by

$$\xi u = zu \times Jv - \zeta u \times Ju, \quad (9.6)$$

$$\xi v = zv \times Jv - \zeta v \times Ju, \quad (9.7)$$

$$\xi z = -\langle u \times Jv, Ju \rangle = \det(u, Ju, Jv), \quad (9.8)$$

$$\xi \zeta = \langle v \times Ju, Jv \rangle = \det(v, Ju, Jv), \quad (9.9)$$

and

$$\eta u = zu \times v, \quad (9.10)$$

$$\eta v = -\zeta v \times u, \quad (9.11)$$

$$\eta z = -\langle u \times v, Ju \rangle = \det(v, u, Ju), \quad (9.12)$$

$$\eta \zeta = \langle v \times u, Jv \rangle = \det(v, u, Jv), \quad (9.13)$$

respectively.

The equations (9.6)—(9.13) define polynomial vector fields ξ and η in \mathbf{C}^8 , homogeneous of degree three. The vector fields ξ and η in \mathbf{C}^8 commute with each other. Finally both vector fields ξ and η are divergence-free, and have the six functions

$$\begin{aligned} f_1 &:= \langle u, u \rangle, & f_2 &:= \langle u, v \rangle, & f_3 &:= \langle v, v \rangle, \\ f_4 &:= \langle u, Ju \rangle + z^2, & f_5 &:= \langle u, Jv \rangle + z\zeta, & f_6 &:= \langle v, Jv \rangle + \zeta^2 \end{aligned} \quad (9.14)$$

as constants of motion. (We have chosen the minus sign in (9.5) in order to get a plus sign in f_5 .) All these statements are true when J , which appears as the parameter in (9.6)—(9.13), is a symmetric 3×3 -matrix.

For the generic values of the f_i , the equations (9.14) define a smooth two-dimensional affine algebraic variety M in \mathbf{C}^8 , on which ξ and η are commuting vector fields. Note that Chaplygin's sphere corresponds to the case that

$$f_1 = 1, f_2 = j_3, f_3 = \|j\|^2, f_4 = 1/\rho, f_5 = 0, f_6 = 2T. \quad (9.15)$$

In contrast with the real case, the complex surface is not compact and therefore we cannot conclude that the flows of ξ and η with complex times lead to an identification of M with a complex torus. Also, the flows of ξ and η on M with complex times will not be complete in the sense that these are not defined for all complex times and therefore do not define an action of \mathbf{C}^2 on M . In Proposition 10.8 we will obtain a completion \hat{M} of M which is isomorphic to a complex torus on which the vector fields ξ and η are constant (and linearly independent). A very different construction, based on Chaplygin's integration of the system in terms of hyperelliptic integrals, will be described in Subsection 11.4.

The system (9.6)—(9.9) and even more so the integrals (9.14) resemble the system (2) and the integrals $\langle X, X \rangle$, (3), (5) and (6) in the article of Adler and van Moerbeke [2]. However, there are also differences: we have the intersection of six quadrics in an eight-dimensional space, whereas in [2] one has the intersection of “only” four quadrics in a six-dimensional space. In Subsection 9.3 we will show that the system (9.6)—(9.9) can be mapped to the geodesic flow on the Euclidean motion group for a left invariant metric, cf. Subsection 9.3. The latter system resembles the ones in the article of Adler and van Moerbeke [2] even more closely, only with the six-dimensional Lie algebra of $\mathrm{SO}(4)$ replaced by the six-dimensional Lie algebra of the Euclidean motion group in the three-dimensional space. Like in [2], the vector field in this Lie algebra is homogeneous of degree two and has four quadratic constants of motion.

9.2 Reduction to Horizontal Moment

In this subsection we assume that the moment j of the momentum around the point of contact is not vertical. We will investigate how the vector field ξ defined by (9.6)—(9.9) changes if we apply a linear substitution of variables of the form

$$u = a\tilde{u} + b\tilde{v} \quad \text{and} \quad v = c\tilde{u} + d\tilde{v}. \quad (9.16)$$

The equations (9.16) are equivalent to $du - bv = D\tilde{u}$, $-cu + av = D\tilde{v}$, in which $D := ad - bc$. Substituting (9.6) and (9.7) in $D\xi\tilde{u} = d\xi u - b\xi v$, $D\xi\tilde{v} = -c\xi u + a\xi v$, we obtain with a straightforward calculation that

$$D\xi\tilde{u} = D(zc - \zeta a)\tilde{u} \times J\tilde{u} + D(zd - \zeta b)\tilde{u} \times J\tilde{v},$$

and in a similar fashion that

$$\xi\tilde{v} = (zc - \zeta a)\tilde{v} \times J\tilde{u} + (zd - \zeta b)\tilde{v} \times J\tilde{v}.$$

These equations are of the form (9.6), (9.7), with u, v, z and ζ replaced by $\tilde{u}, \tilde{v}, \tilde{z}$ and $\tilde{\zeta}$, respectively, if we take

$$\tilde{z} := zd - \zeta b, \quad -\tilde{\zeta} := zc - \zeta a. \quad (9.17)$$

Applying ξ to (9.17) and substituting (9.8), (9.9) and (9.16), we obtain with a straightforward calculation that

$$\xi\tilde{z} = D^2 \det(\tilde{u}, \tilde{J}\tilde{u}, \tilde{J}\tilde{v}), \quad -\xi\tilde{\zeta} = D^2 \det(\tilde{v}, \tilde{J}\tilde{v}, \tilde{J}\tilde{u}).$$

It follows that the vector field ξ is invariant if we arrange that

$$(ad - bc)^2 = D^2 = 1. \quad (9.18)$$

The constants of motion (9.14) are related to the corresponding ones

$$\begin{aligned} \tilde{f}_1 &:= \langle \tilde{u}, \tilde{u} \rangle, & \tilde{f}_2 &:= \langle \tilde{u}, \tilde{v} \rangle, & \tilde{f}_3 &:= \langle \tilde{v}, \tilde{v} \rangle, \\ \tilde{f}_4 &:= \langle \tilde{u}, \tilde{J}\tilde{u} \rangle + \tilde{z}^2, & \tilde{f}_5 &:= \langle \tilde{u}, \tilde{J}\tilde{v} \rangle + \tilde{z}\tilde{\zeta}, & \tilde{f}_6 &:= \langle \tilde{v}, \tilde{J}\tilde{v} \rangle + \tilde{\zeta}^2 \end{aligned} \quad (9.19)$$

with tildes over all the variables, by means of the formulas

$$f_1 = a^2 \tilde{f}_1 + 2ab \tilde{f}_2 + b^2 \tilde{f}_3, \quad (9.20)$$

$$f_2 = ac \tilde{f}_1 + (ad + bc) \tilde{f}_2 + bd \tilde{f}_3, \quad (9.21)$$

$$f_3 = c^2 \tilde{f}_1 + 2cd \tilde{f}_2 + d^2 \tilde{f}_3, \quad (9.22)$$

$$f_4 = a^2 \tilde{f}_4 + 2ab \tilde{f}_5 + b^2 \tilde{f}_6, \quad (9.23)$$

$$f_5 = ac \tilde{f}_4 + (ad + bc) \tilde{f}_5 + bd \tilde{f}_6, \quad (9.24)$$

$$f_6 = c^2 \tilde{f}_4 + 2cd \tilde{f}_5 + d^2 \tilde{f}_6. \quad (9.25)$$

Here we have used in (9.23), (9.24), (9.25) that (9.17) imply that $a\tilde{z} + b\tilde{\zeta} = Dz$ and $c\tilde{z} + d\tilde{\zeta} = D\zeta$, whereas (9.18) implies that $D^{-2} = 1$.

Consider the matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $F = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_3 \end{pmatrix}$, $G = \begin{pmatrix} f_4 & f_5 \\ f_5 & f_6 \end{pmatrix}$. If \tilde{F} is equal to the matrix F with tildes over the coefficients, then the equations (9.20), (9.21), (9.22) are equivalent to the matrix equation $F = M\tilde{F}M^*$. Here M^* denotes the transposed of M . Similarly, if \tilde{G} is equal to the matrix G with tildes over the coefficients, then the

equations (9.23), (9.24), (9.25) are equivalent to the matrix equation $G = M \tilde{G} M^*$. An obvious consequence is that, for any $\lambda \in \mathbf{C}$,

$$p(\lambda) := \det(G - \lambda F) = \det(\tilde{G} - \lambda \tilde{F}). \quad (9.26)$$

Note that $p(\lambda) = \alpha \lambda^2 - \beta \lambda + \gamma$, in which

$$\alpha = \det F = f_1 f_3 - f_2^2, \quad \beta = f_1 f_6 + f_3 f_4 - 2f_2 f_5, \quad \text{and} \quad \gamma = \det G = f_4 f_6 - f_5^2.$$

Therefore the invariance of the polynomial p is equivalent to the three equations $\det F = \det \tilde{F}$, $\det G = \det \tilde{G}$ and $f_1 f_6 + f_3 f_4 - 2f_2 f_5 = \tilde{f}_1 \tilde{f}_6 + \tilde{f}_3 \tilde{f}_4 - 2\tilde{f}_2 \tilde{f}_5$. In the case of Chaplygin's sphere, we have $p(\lambda) = \alpha \lambda^2 - \beta \lambda + \gamma$ with

$$\alpha = \|j\|^2 - j_3^2 = j_1^2 + j_2^2, \quad \beta = 2T + \|j\|^2/\rho, \quad \text{and} \quad \gamma = 2T/\rho,$$

cf. (9.15).

Suppose that all the coefficients are real and that F is positive definite. Then, first diagonalizing F by means of an orthogonal transformation, one subsequently can obtain a real 2×2 -matrix A such that $F = A A^*$. It follows that $(\det A)^2 = \det F > 0$, which implies that A is invertible. There exists an orthogonal transformation O such that $D := O^{-1} G' O = O^{-1} G' (O^{-1})^*$ is diagonal, where G' is equal to the symmetric matrix $A^{-1} G (A^{-1})^*$. In other words, $G = A G' A^* = A O D (A O)^*$. For an arbitrary invertible diagonal matrix B we have now arranged that $G = M \tilde{G} M^*$, with $M = A O B^{-1}$ and $\tilde{G} := B D B^*$ diagonal. In order to arrange that $(\det M)^2 = 1$, it is sufficient to take $1 = (\det A)^2 (\det B)^{-2} = \det F (\det B)^{-2}$, or $(\det B)^2 = \det F$. We now have $F = M \tilde{F} M^*$ with $\tilde{F} = B B^*$. If we choose $B = \text{diag}(1, (\det F)^{1/2})$, then we have arrived at the situation that $\tilde{f}_1 = 1$, $\tilde{f}_2 = 0$, $\tilde{f}_3 = \det F$, $\tilde{f}_5 = 0$. Note also that in O we still have the freedom to precede it by the matrix which switches the two basis vectors, which means that we still can switch \tilde{f}_4 and \tilde{f}_6 .

Remark 9.1 A diagonal matrix remains unchanged if it is multiplied from the left and the right by the matrix $\text{diag}(1, -1)$, which has determinant equal to -1 . Therefore we can arrive at the same diagonal matrices \tilde{F} , \tilde{G} with the help of a matrix M which satisfies the stronger condition $\det M = 1$ instead of $\det M = \pm 1$ as required in (9.18).

In other words, $\text{SL}(2, \mathbf{C})$ acts as a symmetry group for the vector fields ξ and η . The functions f_i are not invariant under the action of $\text{SL}(2, \mathbf{C})$ and actually the $\text{SL}(2, \mathbf{C})$ -action can be used to change to levels \tilde{f}_i such that $\tilde{f}_1 = 1$, $\tilde{f}_2 = 0$, $\tilde{f}_3 = \det F$, $\tilde{f}_5 = 0$. On the other hand, the coefficients of the polynomial p in (9.26) are invariant, which fact can be used in order to determine \tilde{f}_3 , \tilde{f}_4 and \tilde{f}_6 in terms of the f_i .

In Subsection 9.3 we will give a description of the quotient space under the action of $\text{SL}(2, \mathbf{C})$ ⊙

In the case of Chaplygin's sphere, when we have (9.15), we have that $\det F = \|j\|^2 - j_3^2 = j_1^2 + j_2^2 > 0$ because j is not vertical. Therefore F is positive definite and we have a

reduction to the situation that $\tilde{f}_1 = 1$, $\tilde{f}_2 = 0$ and $\tilde{f}_5 = 0$. It follows then from (9.26) that $\tilde{f}_3 = \det F = j_1^2 + j_2^2$, whereas \tilde{f}_4, \tilde{f}_6 satisfy the equations $\tilde{f}_4 \tilde{f}_6 = \det G = 2T/\rho$ and

$$2T + \|j\|^2/\rho = \tilde{f}_6 + (j_1^2 + j_2^2) \tilde{f}_4.$$

These \tilde{f}_i are again the levels of a Chaplygin's sphere, but with j, ρ, T replaced by $\tilde{j}, \tilde{\rho}, \tilde{T}$, respectively. If $\tilde{j}_3 = 0$, which means that *the new moment \tilde{j} is horizontal*, the length of \tilde{j} is equal to the length of the horizontal projection of j , and $\tilde{\rho}$ and \tilde{T} satisfy the equations $\tilde{T}/\tilde{\rho} = T/\rho$ and

$$2\tilde{T} + (j_1^2 + j_2^2)/\tilde{\rho} = 2T + \|j\|^2/\rho.$$

The solution which depends continuously on the parameters and satisfies $\tilde{T} = T$, $\tilde{\rho} = \rho$ when $j_3 = 0$, is given by

$$\tilde{\rho} = \rho \tilde{T}/T \quad \text{and} \quad 2\tilde{T} = T + \|j\|^2/2\rho + \left[(T - \|j\|^2/2\rho)^2 + 2j_3^2 T/\rho \right]^{1/2}. \quad (9.27)$$

Because $\tilde{T} \neq T$ when j is not already horizontal, we have $\tilde{\rho} \neq \rho$. *Because we do not change the matrix $J = (I + \rho)^{-1}$, this means that we have to change the moment of inertia tensor I to the new one $\tilde{I} = I + \rho - \tilde{\rho}$.*

If we allow complex coefficients, then we can arrive at any \tilde{F}, \tilde{G} such that (9.26) holds, provided that the polynomial p is of degree two and has two distinct zeros, cf. Hodge and Pedoe [19, Vol. II, p. 278]. These conditions are equivalent to the conditions that $f_1 f_3 - f_2^2 \neq 0$ and the discriminant

$$\Delta = (f_1 f_6 + f_3 f_4 - 2f_2 f_5)^2 - 4(f_1 f_3 - f_2^2)(f_4 f_6 - f_5^2) \quad (9.28)$$

of p is not equal to zero. Again we can arrange that $\tilde{f}_1 = 1, \tilde{f}_3 = f_1 f_3 - f_2^2$. The new values \tilde{f}_4 and \tilde{f}_6 are determined from the condition that \tilde{f}_4 and \tilde{f}_6/\tilde{f}_3 are equal to the zeros of p , which are unique up to their ordering.

Remark 9.2 In the case of Chaplygin's sphere, when we have (9.15). and similar equations with tildes over all the symbols, the equations (9.20), (9.21), (9.22) are equivalent to the statement that there exists a rotation $C \in \text{SO}(3)$ such that $C e_3 = a e_3 + b \tilde{j}$ and $C j = c e_3 + d \tilde{j}$. These equations are equivalent to (9.16) if $u = A^{-1} e_3$, $v = A^{-1} j$, $\tilde{u} = \tilde{A}^{-1} e_3$ and $\tilde{v} = \tilde{A}^{-1} \tilde{j}$, in which $\tilde{A} = C A$, $A \in \text{SO}(3)$. The invariance of the vector field ξ under the substitutions (9.16) then means that if $A(\tau)$ denotes the rotational motion as a function of the reparametrized time τ , then $\tilde{A}(\tau) = C A(\tau)$ satisfies a differential equation of the same form as $A(\tau)$, but with j, ρ, I and T replaced by $\tilde{j}, \tilde{\rho}, \tilde{I}$ and \tilde{T} , respectively. \otimes

Remark 9.3 If j is not vertical, then it is impossible to make \tilde{j} vertical. Actually the rotational system with nonvertical j can not be transformed in any algebraic fashion to the system with vertical moment of Section 5.

Indeed, if j is vertical, then the completion of the complexification of the phase space of the $\text{SO}(2)$ -reduced system is equal to the elliptic curve defined by (5.9), (5.10). On the other

hand the complexification of the $\mathrm{SO}(2)$ -action is a free action of \mathbf{C}^\times , where \mathbf{C}^\times denotes the multiplicative group of the nonzero complex numbers. (The mapping $t \mapsto e^t$ is an isomorphism from the additive group $\mathbf{C}/2\pi i \mathbf{Z}$ onto the multiplicative group \mathbf{C}^\times .) In this way the completion of the complexification of the phase space of the system for vertical j is a \mathbf{C}^\times -bundle over an elliptic curve, which is not isomorphic to the Jacobi variety $\mathrm{Jac}(C)$ of the hyperelliptic curve C which we obtain when the moment is not vertical. In particular the \mathbf{C}^\times -bundle is not compact because \mathbf{C}^\times is not compact, whereas $\mathrm{Jac}(C)$ is compact.

The passage from nonvertical to vertical j is an example of “a degenerate limit of an abelian variety, as an extension of a power of \mathbf{C}^\times by an abelian variety”, mentioned by Mumford [32, p. 3.53]. \oslash

For the motion of the point of contact p , we observe that our substitutions imply that, with ω as in (3.11), $z\omega = zJv - \zeta Ju = \tilde{z}\tilde{J}\tilde{v} - \tilde{\zeta}\tilde{J}\tilde{u} = \tilde{z}\tilde{\omega}$. Therefore (3.1) yields that

$$\frac{d\langle p, j \rangle}{d\tau} = z r \langle (A\omega) \times e_3, j \rangle = z r \langle \omega \times u, v \rangle = \tilde{z} r D \langle \tilde{\omega} \times \tilde{u}, \tilde{v} \rangle,$$

which is equal to a constant times the same function for the new system with horizontal moment, and therefore equal to a constant times the rational function on the double covering of $\mathrm{Jac}(\tilde{C})$, described after (11.74). Here \tilde{C} denotes the hyperelliptic curve corresponding to the new system with the horizontal moment.

A similar calculations yields for the $j \times e_3$ -component of p that

$$\frac{d\langle p, j \times e_3 \rangle}{d\tau} = D \tilde{z} \left(a 2T + b \tilde{\zeta}/\tilde{\rho} \right).$$

The function \tilde{z} has a similar description as a rational function on a double covering of $\mathrm{Jac}(\tilde{C})$, with $\tilde{\gamma}$ replaced by $-1/\tilde{\rho}$. However, the function $\tilde{z}\tilde{\zeta} = -\langle \tilde{u}, \tilde{J}\tilde{v} \rangle$ does not seem to have an equally straightforward description in terms of $\mathrm{Jac}(\tilde{C})$.

9.3 Geodesic Flow on the Euclidean Motion Group

The form of the equations (9.6)–(9.13) suggests to introduce the vectors

$$q := u \times v \quad \text{and} \quad r := z v - \zeta u. \quad (9.29)$$

The vector $(q, r) \in \mathbf{C}^3 \times \mathbf{C}^3 = \mathbf{C}^6$ represents the exterior product of the vectors (u, z) and (v, ζ) in \mathbf{C}^4 , and therefore the mapping $((u, z), (v, \zeta)) \mapsto (q, r)$ from \mathbf{C}^8 to \mathbf{C}^6 will be denoted by \wedge .

For Chaplygin’s sphere, it follows from (3.8) that

$$q = u \times v = A^{-1} (e_3) \times A^{-1} j = A^{-1} (e_3 \times j),$$

which means that q is equal to the vector $e_3 \times j$ in body coordinates. Furthermore, it follows from (9.2) and (9.5) that $Jr = z\omega$, which is equal to the rotational velocity with respect to the time variable τ which is related to t by $dt/d\tau = X^{1/2} = z$, cf. Corollary 8.4 and (9.1).

We have

$$\xi q = (\xi u) \times v + u \times (\xi v) = (u \times J r) \times v + u \times (v \times J r) = (u \times v) \times J r = q \times J r,$$

in which the third identity follows from the Jacobi identity $(u \times r) \times v + (r \times v) \times u + (v \times u) \times r$ in $\mathfrak{so}(3)$. Similarly

$$\begin{aligned} \xi r &= (\xi z) v - (\xi \zeta) u + z (\xi v) - \zeta (\xi u) \\ &= \langle u, J u \times J v \rangle v - \langle v, J u \times J v \rangle v + z v \times J r - \zeta u \times J r \\ &= (u \times v) \times (J u \times J v) + r \times J r. \end{aligned}$$

For any vector w we have that

$$\langle J u \times J v, J w \rangle = \det(J u, J v, J w) = \det J \det(u, v, w) = \det J \langle u \times v, w \rangle,$$

which in view of the symmetry of J implies that $J(J u \times J v) = \det J u \times v$, or $J u \times J v = \det J J^{-1}(u \times v)$ when J is invertible. It follows that

$$J u \times J v = J^{\text{co}}(u \times v), \quad \text{in which} \quad J^{\text{co}} = \text{diag}(J_2 J_3, J_3 J_1, J_1 J_2). \quad (9.30)$$

Note that $J^{\text{co}} = (\det J) J^{-1}$ when J is invertible, but (9.30) also holds for noninvertible J . This means that the mapping \wedge intertwines the vector field ξ in \mathbf{C}^8 with the vector field ξ in \mathbf{C}^6 defined by

$$\xi q = q \times J r \quad \text{and} \quad \xi r = q \times J^{\text{co}} q + r \times J r. \quad (9.31)$$

Similarly, we have

$$\eta q = (\eta u) \times v + u \times (\eta v) = z q \times v + u \times \zeta q = q \times (z v - \zeta u) = q \times r,$$

and

$$\begin{aligned} \eta r &= (\eta z) v - (\eta \zeta) u + z (\eta v) - \zeta (\eta u) = -\langle J u, q \rangle v + \langle J v, q \rangle u + z v \times r - \zeta u \times r \\ &= -\langle u, J q \rangle v + \langle v, J q \rangle u + r \times r = -(u \times v) \times J q = -q \times J q. \end{aligned}$$

Therefore the mapping \wedge intertwines the vector field η in \mathbf{C}^8 with the vector field η in \mathbf{C}^6 defined by

$$\eta q = q \times r \quad \text{and} \quad \eta r = -q \times J q. \quad (9.32)$$

If q and r is interpreted as a rotational and translational velocity vector, then the (q, r) -space can be identified with the (complexified) Lie algebra $\mathfrak{e}(3)$ of the Euclidean motion group $E(3)$ in the three dimensional Euclidean space, with the Lie brackets defined by

$$[(q, r), (q', r')] = (q \times q', q \times r' + r \times q'). \quad (9.33)$$

Therefore the vector field ξ defined by (9.31) has the form of a Lax pair $dX/d\tau = [X, L_\xi(X)]$, where L_ξ is the linear transformation in $\mathfrak{e}(3)$ defined by

$$L_\xi(q, r) = (J r, J^{\text{co}} q). \quad (9.34)$$

Similarly the vector field η defined by (9.32) has the form of a Lax pair $dX/d\tau = [X, L_\eta(X)]$, where L_η is the linear transformation in $\mathfrak{e}(3)$ defined by

$$L_\eta(q, r) = (r, -Jq). \quad (9.35)$$

The Lax pair form of ξ and η implies that the flows of both vector fields leave the conjugacy classes in $\mathfrak{e}(3)$ invariant. Because the functions

$$h_1(q, r) := \langle q, q \rangle \quad \text{and} \quad h_2(q, r) := \langle q, r \rangle \quad (9.36)$$

are constant on the conjugacy classes, it follows that *both functions h_1 and h_2 are constants of motion for ξ and η .*

Let G be a Lie group and f a function on the cotangent bundle T^*G of G which is invariant under all left multiplications by elements of G . The quotient space of T^*G by means of the left action of G on T^*G is naturally identified with the dual space \mathfrak{g}^* of the Lie algebra \mathfrak{g} of G , and we denote the restriction of f to \mathfrak{g}^* with the same letter. The canonical Poisson structure on T^*G induces a Poisson structure on \mathfrak{g}^* in such a way that the Hamiltonian vector field H_f of f on \mathfrak{g}^* is given by

$$-\langle X, H_f(l) \rangle = \langle [X, df(l)], l \rangle, \quad X \in \mathfrak{g}, \quad l \in \mathfrak{g}^*. \quad (9.37)$$

The vector field H_f is tangent to the coadjoint orbits in \mathfrak{g}^* , on which the Poisson structure is given by a symplectic structure. This means that on each coadjoint orbit the vector field H_f is Hamiltonian with respect to this symplectic structure. This construction has been introduced already by Lie in [25, Kap. 19] under the name “Die dualistische der adjungierte Gruppe”. It has been rediscovered independently by Kostant, Kirillov, and Souriau. The coadjoint orbits with their symplectic structure are the Marsden-Weinstein reduced phase spaces of T^*G for the left action of G on T^*G . See also Abraham and Marsden [1, Sections 4.3, 4.4].

When $\mathfrak{g} = \mathfrak{e}(3)$, cf. (9.33, then the right hand side in (9.37) takes the form

$$\begin{aligned} & \left\langle \left[(q', r'), \left(\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b} \right) \right], (a, b) \right\rangle = \langle q' \times \frac{\partial f}{\partial a}, a \rangle + \langle q' \times \frac{\partial f}{\partial b} + r' \times \frac{\partial f}{\partial a}, b \rangle \\ & = \langle q', \frac{\partial f}{\partial a} \times a + \frac{\partial f}{\partial b} \times b \rangle + \langle r', \frac{\partial f}{\partial a} \times b \rangle. \end{aligned}$$

Therefore, if we identify the point $(a, b) \in \mathfrak{g}^*$ with the point $(q, r) \in \mathfrak{g}$ with $q = b$ and $r = a$, then we recognize from (9.31) that ξ is equal to the Hamiltonian vector field of the function $f = h_3/2$, in which

$$h_3(q, r) := \langle q, J^{\text{co}} q \rangle + \langle r, J r \rangle. \quad (9.38)$$

With the same identifications the vector field η is equal to the Hamiltonian vector field defined by the function

$$h_4(q, r) := -\langle q, J q \rangle + \langle r, r \rangle. \quad (9.39)$$

Remark 9.4 The Lie algebra $\mathfrak{e}(3)$ has not been identified with its dual $\mathfrak{e}(3)^*$ by means of the Killing form $-4\langle q, q \rangle$, which is degenerate, but by means of the nondegenerate invariant quadratic form $h_2 = \langle q, r \rangle$.

The function h_3 defines a left invariant metric on $T^*E(3)$, which can be used to identify $T^*E(3)$ with the tangent bundle of $E(3)$. Under this identification the flow of the Hamiltonian system of $h_3/2$ corresponds to the geodesic flow on $TE(3)$ defined by the dual metric on $TE(3)$.

We therefore obtain the following, somewhat roundabout correspondence between this geodesic flow and our vector field ξ . First pass from the geodesic flow of the left invariant metric on the tangent bundle to the Hamiltonian system of the function $h_3/2$ on the cotangent bundle, using the metric in order to identify the tangent bundle with the cotangent bundle. Then pass to the reduced system on $\mathfrak{e}(3)^*$ by means of the left action of $E(3)$ on $T^*E(3)$. In the next step, pass to the vector field ξ on $\mathfrak{e}(3)$ using the identification of $\mathfrak{e}(3)$ with $\mathfrak{e}(3)^*$ by means of the quadratic form $\langle q, r \rangle$. Finally the mapping \wedge intertwines the vector field ξ in the $((u, z), (v, \zeta))$ -space with the vector field ξ in the (q, r) -space $\mathfrak{e}(3)$. \odot

The Poisson brackets $\{f, g\} := H_f g$ define a Lie algebra structure on the space of functions. In particular it is antisymmetric, which implies that $H_f f = 0$ and $H_f g = 0$ if and only if $H_g f = 0$. It follows from $H_f f = 0$ that h_3 and h_4 is a constant of motion for ξ and η , respectively. Moreover,

$$\eta h_3/2 = \langle \eta q, J^{\text{co}} q \rangle + \langle \eta r, J r \rangle = \langle q \times r, J^{\text{co}} q \rangle - \langle q \times J q, J r \rangle = 0,$$

because

$$\langle q \times J q, J r \rangle = \langle q, J q \times J r \rangle = \langle q, J^{\text{co}}(q \times r) \rangle = \langle J^{\text{co}} q, q \times r \rangle,$$

cf. (9.30). This implies that h_3 and h_4 are constants of motion for both vector fields ξ and η . The Jacobi identity of the Poisson structure implies that $[H_f, H_g] = H_{\{f, g\}}$. Because we just proved that $\{h_3, h_4\} = 0$, it follows also that the vector fields ξ and η commute.

In particular the system in $\mathfrak{e}(3) \simeq \mathbf{C}^6$ defined by the vector field ξ is completely integrable, as a Hamiltonian system on the four-dimensional coadjoint orbits determined by fixing the values of the functions h_1 and h_2 , defined in (9.36). The Hamiltonian function is the function $h_3/2$ with h_3 defined in (9.38) and the function h_4 defined in (9.39) is the additional constant of motion which yields the complete integrability of the Hamiltonian system.

Remark 9.5 The system in the (q, r) -space resembles the one in the paper of Adler and van Moerbeke [2] very closely: it is defined on a six-dimensional (dual of a) Lie algebra ($\mathfrak{e}(3)$ instead of the Lie algebra $\mathfrak{so}(4)$ of [2]) and has four quadratic constants of motion. Furthermore it is Hamiltonian on coadjoint orbits, and we have two polynomial vector fields which are homogeneous of degree two. In this respect the vector fields are simpler than the vector fields ξ and η defined in (9.6)—(9.13).

Also the behaviour at infinity the level surface of the h_i (with $h_2 = 0$) in the complex projective space is very similar to the behaviour of the level surface in $\mathfrak{so}(4)$ as described in Mumford's appendix to [2]. See Subsection 10.5.

On the other hand it turns out that this behaviour is more singular than that of the projective closure of the level surface M of the functions (9.14), see Subsection 10.5. For this reason we start Section 10 with a discussion of the system in \mathbf{C}^8 , because this seemingly more complicated system has a simpler behaviour at infinity. \odot

We now turn to a closer examination of the mapping \wedge from \mathbf{C}^8 to \mathbf{C}^6 defined by (9.29). To begin with, if (q, r) belongs to the image of \wedge , then $\langle q, r \rangle = 0$, which means that \wedge is a mapping from \mathbf{C}^8 to the hypersurface $h_2 = 0$ in \mathbf{C}^6 , cf. (9.36). Conversely, any $q \in \mathbf{C}^3$ can be written as $u \times v$ for some $u, v \in \mathbf{C}^3$. If $q \neq 0$, then u and v are linearly independent and span the orthogonal complement of q , which implies that for every $r \in \mathbf{C}^3$ such that $\langle q, r \rangle = 0$ there exist $z, \zeta \in \mathbf{C}$ such that $r = zv - \zeta u$. If $q = 0$, then we have (9.36) for $z = 1, v = r, \zeta = 0, u = 0$. It follows that \wedge is surjective from \mathbf{C}^8 onto the hypersurface $h_2 = 0$ in \mathbf{C}^6 .

Remark 9.6 If $q \neq 0$, then $\langle q, r \rangle = 0$ if and only if there exists a vector x_0 such that $q \times x_0 + r = 0$, and every vector x such that $q \times x + r = 0$ is of the form $x = x_0 + c q$ for some scalar c . Therefore the condition that $\langle q, r \rangle = 0$ means that the infinitesimal motion $(q, r) \in \mathfrak{e}(3)$ is either equal to an infinitesimal translation ($q = 0$), or to an infinitesimal rotation about some axis in the three-dimensional case: “no spiralling”. \odot

If we interpret (q, r) as an element of $\bigwedge^2 \mathbf{C}^4$, then the condition $\langle q, r \rangle = 0$ means that the rank of (q, r) is smaller than four. If $(q, r) \neq (0, 0)$, then its null space is two-dimensional and is spanned by (u, z) and (v, ζ) if $(q, r) = (u, z) \wedge (v, \zeta) = (u \times v, zv - \zeta u)$. Note that (u, z) and (v, ζ) are linearly independent in this case. If $(q, r) = (u', z') \wedge (v', \zeta')$, then (u', z') and (v', ζ') are also contained in the null space of (q, r) , and therefore there are unique $a, b, c, d \in \mathbf{C}$ such that

$$(u', z') = a(u, z) + b(v, \zeta) \quad \text{and} \quad (v', \zeta') = c(u, z) + d(v, \zeta), \quad (9.40)$$

and for such vectors (u', z') and (v', ζ') we have that $(q, r) = (u', z') \wedge (v', \zeta')$ if and only if $ad - bc = 1$. In other words, if $\langle q, r \rangle = 0$ and $(q, r) \neq (0, 0)$, then the fiber of (q, r) for the mapping \wedge is equal to the orbit in the $((u, z), (v, \zeta))$ -space \mathbf{C}^8 of the action of $\text{SL}(2, \mathbf{C})$ defined by (9.40).

Let U denote the set of the $((u, z), (v, \zeta)) \in \mathbf{C}^8$ such that the vectors (u, z) and (v, ζ) in \mathbf{C}^4 are linearly independent, and let V be the set of $(q, r) \in \mathbf{C}^6$ such that $\langle q, r \rangle = 0$ and $(q, r) \neq (0, 0)$. Then U is an open subset of \mathbf{C}^8 and V is an open subset, equal to the smooth part, of the 5-dimensional hypersurface $\langle q, r \rangle = 0$ in \mathbf{C}^6 . The action of $\text{SL}(2, \mathbf{C})$ on U is free and the mapping \wedge identifies V with the orbit space of the $\text{SL}(2, \mathbf{C})$ -action on U .

In Subsection 9.2 we had observed that the action of $\text{SL}(2, \mathbf{C})$ leaves the vector fields ξ and η invariant, cf. Remark 9.1. Therefore the projection $\wedge : U \rightarrow V$ intertwines the vector fields ξ and η on U with uniquely defined vector fields on V , which we denoted with the same letters. The fact that the vector fields ξ and η in U commute implies that their push-forwards under \wedge , the vector fields ξ and η in V , commute as well. In the beginning of this subsection we showed that the vector fields ξ and η in \mathbf{C}^6 defined by (9.31) and (9.32)

extend the vector fields ξ and η in V . The fact that the vector fields ξ and η in \mathbf{C}^6 commute is stronger than the fact that their restrictions to V commute.

As observed in Remark 9.1, the functions f_i are not invariant under the $\text{SL}(2, \mathbf{C})$ -action. However, the coefficients of the polynomial p in (9.26) are invariant under the $\text{SL}(2, \mathbf{C})$ -action, which means that these coefficients can be written as functions of $(q, r) \in V$. Actually, we have:

$$h_1 \circ \wedge = f_1 f_3 - f_2^2, \quad (9.41)$$

$$h_2 \circ \wedge = 0, \quad (9.42)$$

$$h_3 \circ \wedge = f_4 f_6 - f_5^2, \quad (9.43)$$

$$(\text{trace } J) h_1 \circ \wedge + h_4 \circ \wedge = f_1 f_6 + f_3 f_4 - 2f_2 f_5, \quad (9.44)$$

from which the compositions $h_i \circ \wedge$ of the functions h_i with the mapping \wedge can be determined in terms of the functions f_i defined in (9.14). Here the functions h_i are defined by (9.36), (9.38) and (9.39).

Proof The equation (9.41) follows from

$$\det F = f_1 f_3 - f_2^2 = \langle u, u \rangle \cdot \langle v, v \rangle - \langle u, v \rangle^2 = \langle u \times v, u \times v \rangle = \langle q, q \rangle = h_1.$$

For (9.43) we write

$$\begin{aligned} \det G &= f_4 f_6 - f_5^2 = (\langle u, Ju \rangle + z^2) (\langle v, Jv \rangle + \zeta^2) - (\langle u, Jv \rangle + z\zeta)^2 \\ &= \langle u, Ju \rangle \cdot \langle v, Jv \rangle - \langle u, Jv \rangle^2 + z^2 \langle v, Jv \rangle + \zeta^2 \langle u, Ju \rangle - 2z\zeta \langle u, Jv \rangle \\ &= \langle u, \langle v, Jv \rangle Ju - \langle v, Ju \rangle Jv \rangle + \langle zv - \zeta u, J(zv - \zeta u) \rangle \\ &= \langle u, v \times (Ju \times Jv) \rangle + \langle r, Jr \rangle = \langle q, J^{\text{co}} q \rangle + \langle r, Jr \rangle = h_3, \end{aligned}$$

where we used (9.30) in the fifth identity.

Finally, we have that $f_1 f_6 + f_3 f_4 - 2f_2 f_5$ is equal to

$$\langle u, u \rangle (\langle v, Jv \rangle + \zeta^2) + \langle v, v \rangle (\langle u, Ju \rangle + z^2) - 2\langle u, v \rangle (\langle u, Jv \rangle + z\zeta),$$

for the computation of which we write

$$\langle u, u \rangle \cdot \langle v, Jv \rangle + \langle v, v \rangle \cdot \langle u, Ju \rangle - 2\langle u, v \rangle \cdot \langle u, Jv \rangle = \langle u, a \rangle,$$

in which

$$\begin{aligned} a &= \langle v, Jv \rangle u + \langle v, v \rangle Ju - 2\langle u, Jv \rangle v \\ &= \langle v, Jv \rangle u - \langle u, Jv \rangle v + \langle v, v \rangle Ju - \langle v, Ju \rangle v \\ &= Jv \times (u \times v) + v \times (Ju \times v). \end{aligned}$$

It follows that $\langle u, a \rangle = \langle u \times v, b \rangle$, in which $b = u \times Jv + Ju \times v$. Now we have, for any vector c ,

$$\begin{aligned} \langle b, c \rangle &= \det(u, Jv, c) + \det(Ju, v, c) = (\text{trace } J) \det(u, v, c) - \det(u, v, Jc) \\ &= (\text{trace } J) \langle u \times v, c \rangle - \langle u \times v, Jc \rangle = \langle (\text{trace } J - J)(u \times v), c \rangle, \end{aligned}$$

which implies that $b = (\text{trace } J - J)(u \times v)$. Collecting all the results, we arrive at

$$f_1 f_6 + f_3 f_4 - 2f_2 f_5 = \langle q, (\text{trace } J - J)(q) \rangle + \langle r, r \rangle = (\text{trace } J) h_1 + h_4,$$

from which (9.44) follows. \square

In the case of Chaplygin's sphere, we can insert the values (9.15) of the functions f_i , which leads to the values

$$h_1 = j_1^2 + j_2^2, \quad h_2 = 0, \quad h_3 = 2T/\rho, \quad (\text{trace } J) h_1 + h_4 = 2T + \|j\|^2/\rho \quad (9.45)$$

for the constants of motion h_i of the system in the six-dimensional (q, r) -space.

The action of the matrix $M \in \text{SL}(2, \mathbf{C})$ with the coefficients a, b, c, d leaves a given level surface of the functions f_i invariant if and only if, in the notation of Subsection 9.2, we have that $M F M^* = F$ and $M G M^* = G$. Assume that $\det F \neq 0$ and that F and G are not proportional. Then we obtained in Subsection 9.2 that there exists an $A \in \text{SL}(2, \mathbf{C})$ such that the matrices $\tilde{F} = A F A^*$ and $\tilde{G} = A G A^*$ are diagonal, and it follows that \tilde{F} and \tilde{G} are not proportional. With the notation $\tilde{M} = A M A^{-1}$, we now have $\tilde{M} \tilde{F} \tilde{M}^* = \tilde{F}$ and $\tilde{M} \tilde{G} \tilde{M}^* = \tilde{G}$. A straightforward calculation, in which we use that the diagonal matrices \tilde{F} and \tilde{G} are not proportional, leads to the conclusion that $\tilde{M} = \pm 1$, which in turn implies that $M = \pm 1$. If $M = -1$ then it acts on the $((u, z), (v, \zeta))$ -space as the antipodal map $x \mapsto -x$.

It follows that the restriction of the mapping \wedge to the level surface M of the function f_i , where we assume that $\det F \neq 0$ and F and G are not proportional, defines a twofold unbranched covering from M onto the level surface of the functions h_i , for the levels given by (9.41)–(9.44). The fibers of $\wedge|_M$ are pairs of antipodal points, and therefore the mapping \wedge leads to an identification of the level surface $\wedge(M)$ on $h_2 = 0$ of the functions h_1, h_3, h_4 with the quotient $M/\pm 1$ of the surface M by means of the antipodal mapping.

Because the commutation of vector fields is a local property, the fact that the vector fields ξ and η on $\wedge(M)$ commute implies, together with the fact that $\wedge : M \rightarrow \wedge(M)$ is a covering, that the vector fields ξ and η on M commute. This leads to a proof of Corollary 8.4 which is based on the facts that ξ is a Hamiltonian vector field on a coadjoint orbit and has the functions f_i as constants of motion.

9.4 Symmetric Matrices

The equations of motion (3.2), in which $\omega = \omega_j(A)$ is given in terms of A by (2.20), (2.19), (2.12), can be entirely expressed in terms of the (positive definite) symmetric matrix

$$B := A(I + \rho)^{-1} A^{-1}. \quad (9.46)$$

We have

$$\frac{dB}{dt} = [\xi(B)_{\text{op}}, B] := \xi(B)_{\text{op}} \circ B - B \circ \xi(B)_{\text{op}}. \quad (9.47)$$

Here $\xi(B)_{\text{op}}$ denotes the antisymmetric linear mapping $\nu \mapsto \xi(B) \times \nu$ and

$$\xi(B) := A \omega_j(A) = B j + \frac{\rho \langle B j, e_3 \rangle}{1 - \rho \langle B e_3, e_3 \rangle} B e_3. \quad (9.48)$$

The velocity (3.1) of the point of contact p also is a function of B :

$$\frac{dp}{dt} = r \xi(B) \times e_3. \quad (9.49)$$

The equation (9.47) is a Lax system, cf. [24], and therefore the eigenvalues of B are constants of motion. More explicitly, it follows directly from (9.46) that we have the following three constants of motion in the six-dimensional vector space of the symmetric 3×3 -matrices:

$$\text{trace } B^i = \text{trace } (I + \rho)^{-i}, \quad i = 1, 2, 3. \quad (9.50)$$

These equations are homogeneous of degree one, two and three, respectively. The kinetic energy equation (3.14) is a polynomial equation of degree two in B :

$$\langle B j, e_3 \rangle^2 - (\rho^{-1} - \langle B e_3, e_3 \rangle) (2T - \langle B j, j \rangle) = 0. \quad (9.51)$$

When the moment of inertia tensor I , assumed to be diagonal, has three different eigenvalues, then the mapping $A \mapsto B = A(I + \rho)^{-1} A^{-1}$ is a fourfold covering from $\text{SO}(3)$ onto the manifold determined by (9.50), where A and A' are mapped to the same symmetric matrix if and only $A' = A \circ R$ in which R is one of the four diagonal rotations. Apparently the reduction of this symmetry group leads to quite a reduction of the degrees of the constants of motion. After passing to a covering on which the vector field X is single valued and regular, Proposition 11.6 would lead to a mapping to a fourfold covering of the Jacobi variety of a hyperelliptic curve of genus two, on which X corresponds to a constant vector field.

The system (9.47) has a strong resemblance to the equations of van Moerbeke [28, formula (17)] in the lowest dimensional case $N = 3$. In his case the kinetic energy equation (9.51) is replaced by the condition that the “modulus”, the product of the upper triangular elements of B is kept constant. His equations of motion also lead to a constant vector field on the Jacobi variety of a hyperelliptic curve of genus two.

9.5 Chaplygin

Subsection 9.2 reflects our understanding of Chaplygin’s [9, §4]. His $\lambda, \lambda', \mu, \mu'$ in (35) correspond to our a, b, c, d in (9.16). The equations in his (37) correspond to our (9.20), (9.21), (9.22) in which the f_i and \tilde{f}_i are given by (9.15) and the same formulas with tildes over all the variables. The equation $lD\lambda\lambda' + \mu\mu' = 0$ after Chaplygin’s formula (40) corresponds to our (9.24) with $f_5 = \tilde{f}_5 = 0$.

The sentence “The sphere rolls in a direction perpendicular ...” in front of [9, (47)] has not been formulated very accurately. The velocity of the point of contact is neither exactly orthogonal to the moment, nor is it a periodic function of (the reparametrized) time.

The formula [9, (47)] corresponds to our description of $d\langle p, j \rangle / d\tau$. The variable ξ in [9, §4] corresponds to our $\langle p, j \times e_3 \rangle$, but our equations for it differ from the equations which Chaplygin obtained for it at the end of [9, §4]. In the very last formula in [9, §4], we believe that the right hand side has to be replaced by its primitive with respect to the time t , an expression which is not much more transparent than our constant times the primitive with respect to τ of $\tilde{z}\tilde{\zeta}$.

10 Complexification and Completion

10.1 A Smooth Complex Surface

Let $M = M(c)$ denote the set of the solutions $((u, z), (v, \zeta)) \in \mathbf{C}^8$ of the equations $f_i = c_i$, in which f_i are the quadratic forms defined in (9.14). In this subsection we will assume that J is a diagonal matrix with three different eigenvalues J_1, J_2, J_3 on the diagonal. We will furthermore assume that the constants c_i satisfy

$$c_1 c_3 \neq 0, c_2 = 0, c_5 = 0, c_1 c_6 \neq c_3 c_4, (c_4 - J_i c_1) (c_6 - J_i c_3) \neq 0 \text{ for } i = 1, 2 \text{ and } 3. \quad (10.1)$$

As we have seen in Subsection 9.2, we can arrive at the first three conditions $c_1 c_3 \neq 0$, $c_2 = 0$, $c_5 = 0$ and $c_1 c_6 \neq c_3 c_4$, if and only if, in the original system, $\det F = f_1 f_3 - f_2^2 \neq 0$ and the discriminant Δ of the polynomial $p : \lambda \mapsto \det(G - \lambda F)$, given by (9.28), is not equal to zero. Because the polynomial p is invariant under the transformations in Subsection 9.2, the last condition means that none of the J_i is a zero of p . Summarizing, the conditions mean for the original system that the polynomial p is of second order and has two distinct zeros, none of these equal to one of the J_i 's. In Subsection 9.2 we actually arranged also that $c_1 = 1$, as one always has for Chaplygin's sphere.

In the case of Chaplygin's sphere, where we have (9.15) and $J_i = 1/(I_i + \rho)$, these assumptions mean the following.

- i) The J_i are different: there are three different principal moments of inertia I_i .
- ii) $f_1 f_3 - f_2^2 = \|j\|^2 - j_3^2 = j_1^2 + j_2^2 \neq 0$: the moment vector j is not vertical.
- iii) $\Delta = (2T - \|j\|^2/\rho)^2 + 8j_3^2 T/\rho \neq 0$: this is automatically true when $j_3 \neq 0$, or j is not horizontal, because this implies that $j \neq 0$, which in turn implies that $T > 0$. When j is horizontal, then the critical energy levels are equal to $T_{\text{crit}, i} = \|j\|^2/2(I_i + \rho) < \|j\|^2/2\rho$, cf. (4.9). Because T is less than or equal to the largest critical energy level, we have that $2T < \|j\|^2/2\rho$ when $j_3 = 0$, and therefore we have always that $\Delta > 0$.
- iv) For Chaplygin's sphere, we have that

$$p(\lambda) = (\|j\|^2 - j_3^2) \lambda^2 - (2T + \|j\|^2/\rho) \lambda + 2T/\rho.$$

The equation (4.9) for $\iota = I_i$, $J_i = 1/(I_i + \rho)$, turns out to be equivalent to $p(J_i) = 0$ when $T = T_{\text{crit}, i}$. Therefore the last condition in (10.1) follows from the condition that

T is not equal to one of the critical energy levels, or that the real part of the level set is a smooth two-dimensional manifold.

After the reduction to the situation that the moment j is a nonzero horizontal vector, we also have that $c_2 = 0$, whereas $c_5 = 0$ always holds for Chaplygin's sphere. We conclude that for Chaplygin's sphere the conditions are satisfied if the moment vector j is not vertical and T is in between the critical energy levels.

Proposition 10.1 *The derivatives df_i of the functions f_i defined in (9.14) are linearly independent at each point of M , and therefore M is a smooth complex two-dimensional affine algebraic variety.*

Proof We have to prove that if $((u, z), (v, \zeta)) \in M$ and α_i , $1 \leq i \leq 6$ are constants such that $\sum_{i=1}^6 \alpha_i df_i = 0$, then all the α_i are equal to zero. The equations for the α_i amount to

$$2\alpha_1 u + \alpha_2 v + 2\alpha_4 Ju + \alpha_5 Jv = 0, \quad (10.2)$$

$$2\alpha_4 z + \alpha_5 \zeta = 0, \quad (10.3)$$

$$\alpha_2 u + 2\alpha_3 v + \alpha_5 Ju + 2\alpha_6 Jv = 0 \quad \text{and} \quad (10.4)$$

$$\alpha_5 z + 2\alpha_6 \zeta = 0, \quad (10.5)$$

corresponding to the derivatives with respect to u , z , v and ζ , respectively.

If we take the inner product of (10.2) with v , we obtain that

$$0 = \alpha_2 c_3 - 2\alpha_4 z \zeta + \alpha_5 (c_6 - \zeta^2) = \alpha_2 c_3 + \alpha_5 c_6,$$

where we have used $\langle v, u \rangle = 0$, $\langle v, v \rangle = c_3$, $\langle v, Jv \rangle = -z \zeta$, $\langle v, Ju \rangle = c_6 - \zeta^2$, and (10.3). If we take the inner product of (10.4) with u , we obtain that

$$0 = \alpha_2 c_1 + \alpha_5 (c_4 - z^2) - 2\alpha_6 z \zeta = \alpha_2 c_1 + \alpha_5 c_4,$$

where we have used $\langle u, u \rangle = 1$, $\langle u, v \rangle = 0$, $\langle u, Jv \rangle = -z \zeta$, $\langle u, Ju \rangle = c_4 - z^2$, and (10.5). These two equations for α_2 and α_5 lead in combination with $c_3 c_4 \neq c_1 c_6$ to the conclusion that $\alpha_2 = \alpha_5 = 0$.

If we substitute this in (10.2) then it follows that, unless $\alpha_1 = \alpha_4 = 0$, the vectors Ju and u are linearly dependent, which in turn implies that $Ju = J_i u$ for some $i = 1, 2, 3$. It follows that $\langle u, Ju \rangle = J_i \langle u, u \rangle = J_i c_1$, and therefore

$$z^2 = c_4 - J_i c_1 \neq 0. \quad (10.6)$$

From (10.3) with $\alpha_5 = 0$ we obtain that $\alpha_4 z = 0$, which in view of (10.6) implies that $\alpha_4 = 0$. Now (10.2) is equivalent to $\alpha_1 u = 0$, which implies that $\alpha_1 = 0$ because $\langle u, u \rangle = c_1 \neq 0$ implies that $u \neq 0$.

Unless $\alpha_3 = \alpha_6 = 0$, it follows from (10.4), in which $\alpha_2 = \alpha_5 = 0$, that Jv and v are linearly dependent, which implies that $Jv = J_i v$ for some $i = 1, 2, 3$. It follows that $\langle v, Jv \rangle = J_i \langle v, v \rangle = J_i c_3$, and therefore

$$\zeta^2 = c_6 - J_i c_3 \neq 0. \quad (10.7)$$

From (10.5) with $\alpha_5 = 0$ we obtain that $\alpha_6 \zeta = 0$, which in view of (10.7) implies that $\alpha_6 = 0$. Now (10.4) is equivalent to $\alpha_3 v = 0$, which implies that $\alpha_3 = 0$ because $\langle v, v \rangle = c_3 \neq 0$ implies that $v \neq 0$. \square

Proposition 10.2 *The polynomial vector fields ξ and η are linearly independent at every point of M .*

Proof If $u \times Ju = 0$, which means that $u = u_i e_i$ for $i = 1, 2$ or 3 . This implies (10.6) and therefore $z \neq 0$. Because $\langle u, v \rangle = 0$, it follows that the vectors $\xi u = zu \times Jv$ and $\eta u = zu \times v$ can only be linearly dependent if v and Jv are linearly dependent, which implies that $v = v_h e_h$ for some $h = 1, 2$ or 3 . This implies (10.7) with i replaced by h and therefore $\zeta \neq 0$. On the other hand we have that $u_i^2 = \langle u, u \rangle = c_1 \neq 0$, $0 = \langle u, v \rangle = u_i v_i$ hence $v_i = 0$ and therefore $-z\zeta = \langle u, Jv \rangle = u_i J_i v_i = 0$, which leads to a contradiction. In a similar way we obtain that ξ and η are linearly independent when $v \times Jv = 0$.

In the sequel of the proof we therefore may assume that u and Ju are linearly independent and that v and Jv are linearly independent. Assume that $z\zeta = 0$, which in turn implies that $\langle u, Jv \rangle = -z\zeta = 0$, and therefore $(u \times Jv) \times u = c_1 Jv$, whereas $(u \times v) \times v = c_1 v$ because $\langle u, v \rangle = 0$. It follows that $u \times Jv$ and $u \times v$ are linearly independent. If $z \neq 0$ then $\zeta = 0$ and we obtain that $\xi u = zu \times Jv$ and $\eta u = zu \times v$ are linearly independent. If $\zeta \neq 0$ then $z = 0$ and we obtain that $\xi v = -\zeta v \times Ju$ and $\eta v = -\zeta v \times u$ are linearly independent.

If $z = \zeta = 0$, then it follows from $0 = -z\zeta = \langle u, Jv \rangle = \langle Ju, v \rangle$ and $\langle u, v \rangle = 0$ that there are nonzero $a, b \in \mathbf{C}$ such that $u = av \times Jv$ and $v = bu \times Ju$. Inserting this in (9.8), (9.9), (9.12), (9.13), we obtain that

$$\begin{aligned} b\xi z &= \langle v, Jv \rangle = c_6 - \zeta^2 = c_6, \\ -a\xi\zeta &= \langle u, Ju \rangle = c_4 - z\zeta = c_4, \\ b\eta z &= \langle v, v \rangle = c_3, \quad \text{and} \\ -a\eta\zeta &= \langle u, u \rangle = c_1. \end{aligned}$$

It follows that

$$-ab [(\xi z)(\eta\zeta) - (\xi\zeta)(\eta z)] = c_1 c_6 - c_3 c_4 \neq 0,$$

cf. (10.1), which implies that ξ and η are linearly independent.

Finally, suppose that $u \times Ju \neq 0$, $v \times Jv \neq 0$, $z \neq 0$, $\zeta \neq 0$, and $\alpha\xi + \beta\eta = 0$ for some $\alpha, \beta \in \mathbf{C}$. It follows that there are $\gamma, \delta \in \mathbf{C}$ such that

$$\alpha(zJv - \zeta Ju) + \beta zv = \gamma u \quad \text{and} \quad (10.8)$$

$$\alpha(zJv - \zeta Ju) - \beta\zeta u = \delta v. \quad (10.9)$$

If we take the inner product of (10.8) with v , then we obtain that

$$0 = \alpha(z[c_6 - \zeta^2] + \zeta z\zeta) + \beta z c_3 = z(\alpha c_6 + \beta c_3),$$

where we have used that $\langle v, u \rangle = 0$, $\langle v, Jv \rangle = c_6 - \zeta^2$, $\langle v, Ju \rangle = -z\zeta$, and $\langle v, v \rangle = c_3$. Similarly the inner product of (10.9) with u yields

$$0 = \alpha (-z^2 \zeta - \zeta [c_4 - z^2]) - \beta \zeta c_1 = -\zeta (\alpha c_4 + \beta c_1).$$

Because $c_6 c_1 \neq c_4 c_3$, $z \neq 0$ and $\zeta \neq 0$, these two equations for α and β imply that $\alpha = \beta = 0$. \square

Let L denote the manifold of the solutions $(u, v) \in \mathbf{C}^3 \times \mathbf{C}^3$ of (3.9) and (3.14). The projection $((u, z), (v, \zeta)) \mapsto (u, v)$ exhibits M as a two-fold covering of L and intertwines the vector fields η and η defined by (9.6)—(9.13) with the previously defined unoriented vector fields $\xi = X(u)^{1/2} R_\omega$ and $\eta = X(u)^{1/2} R_{(I+\rho)\omega}$ on M_0 , where the word “unoriented” refers to the fact that the latter vector fields are only determined up to their signs.

10.2 At Infinity

The (homogeneous) polynomial nature of the vector fields ξ and η , given by (9.6)—(9.9) and (9.10)—(9.13), respectively, and of the constants of motion (9.14) makes it natural to investigate the system in the eight-dimensional complex projective space \mathbf{CP}^8 . The complex projective space is obtained by adding one more variable, which we denote by ϵ , and then taking the quotient of $\mathbf{C}^9 \setminus \{0\}$ with respect to the actions $x \mapsto cx$ of the multiplicative group \mathbf{C}^\times of the nonzero complex numbers c . The standard coordinate charts correspond to the quotients of the sets of x for which one of the coordinates, say x_i , is nonzero, and then the coordinates for this chart are obtained by putting $x_i = 1$ and using the x_j with $j \neq i$ as the coordinates. The changes of coordinates are obtained by using the identification of x with cx . Although it would be clearer to do so, we will not introduce different notations for the coordinates in the various charts, in order to avoid heavy notations. If we put $\epsilon = 1$, then we obtain the affine space \mathbf{C}^8 as a subset, equal to one of the standard coordinate charts, of \mathbf{CP}^8 . The complement $\mathbf{CP}_\infty^8 := \mathbf{CP}^8 \setminus \mathbf{C}^8$ corresponds to taking $\epsilon = 0$ in the other coordinate charts, in which one of the coordinates of $((u, z), (v, \zeta))$ is taken equal to 1. In this way $\mathbf{CP}^8 \setminus \mathbf{C}^8$ is identified with \mathbf{CP}^7 . We will refer to $\mathbf{C}^8 \subset \mathbf{CP}^8$ and $\mathbf{CP}_\infty^8 \simeq \mathbf{CP}^7$ as the affine (or finite) part of \mathbf{CP}^8 and the projective space at infinity, respectively.

For any choice of the constants c_i , we denote by $M(c)$ the set of the solutions in \mathbf{C}^8 of the equations $f_i = c_i$, the level set of the constants of motion. As a subset of \mathbf{CP}^8 , the set $M(c)$ is obtained by homogenizing the equations. In view of the homogeneity of the f_i of degree two, this corresponds to replacing the equations $f_i = c_i$ by

$$g_i((u, z), (v, \zeta), \epsilon) := f_i((u, z), (v, \zeta)) - c_i \epsilon^2 = 0. \quad (10.10)$$

Let $N(c)$ denote the set of solutions of (10.10) in \mathbf{CP}^8 . Note that $M(c) = N(c) \cap \mathbf{C}^8 = N(c) \setminus \mathbf{CP}_\infty^8$ is equal to the affine part of $N(c)$.

Our goal in this section is to study the closure $\overline{M(c)}$ of $M(c)$ in \mathbf{CP}^8 , especially in the case that $M(c) = M$ with c and M as in Subsection 10.1. Here the closure is taken with

respect to the ordinary topology, but it is known that $\overline{M(c)}$ is equal to a projective algebraic variety, and therefore also closed in the Zariski topology. See Łojasiewicz [26, p. 383].

The solutions of the equations (10.10) at \mathbf{CP}_∞^8 are obtained by putting $\epsilon = 0$ in (10.10), in which case we obtain the equations

$$\begin{aligned} \langle u, u \rangle &= 0, & \langle u, v \rangle &= 0, & \langle v, v \rangle &= 0, \\ \langle u, Ju \rangle + z^2 &= 0, & \langle u, Jv \rangle + z\zeta &= 0, & \langle v, Jv \rangle + \zeta^2 &= 0. \end{aligned} \quad (10.11)$$

The solutions of (10.11) in \mathbf{C}^8 form the conic affine algebraic variety $M(0)$, and the corresponding projective variety in \mathbf{CP}^7 , which we denote by $M(0)_\infty$, is equal to the set of the solutions in \mathbf{CP}_∞^8 of the equations (10.10). Note that $M(0)_\infty = N(c) \cap \mathbf{CP}_\infty^8$ does not depend on the choice of the c_i .

Lemma 10.3 *The set $M(0)$ is a three-dimensional conic subvariety of \mathbf{C}^8 , consisting of the $((u, z), (v, \zeta))$ such that $\langle u, u \rangle = 0$, $\langle u, Ju \rangle + z^2 = 0$ and the vectors (u, z) and (v, ζ) in \mathbf{C}^4 are linearly dependent. If $u = 0$ and $z = 0$, then we have to add the conditions that $\langle v, v \rangle = 0$ and $\langle v, Jv \rangle + \zeta^2 = 0$. The corresponding projective variety $M(0)_\infty = N(c) \cap \mathbf{CP}_\infty^8$ at infinity is a smooth two-dimensional subvariety of \mathbf{CP}_∞^8 .*

Proof If $u \neq 0$, then $\langle u, v \rangle = 0$ implies that $v = u \times w$ for some $w \in \mathbf{C}^3$. Using $\langle u, u \rangle = 0$, we obtain that

$$0 = \langle v, v \rangle = \langle u \times w, u \times w \rangle = \langle u, u \rangle \cdot \langle w, w \rangle - \langle u, w \rangle^2 = -\langle u, w \rangle^2,$$

or $\langle u, w \rangle = 0$, which in turn implies that $w = u \times a$ for some $a \in \mathbf{C}^3$. But then

$$v = u \times (u \times a) = \langle u, a \rangle u - \langle u, u \rangle a = \langle u, a \rangle u,$$

which shows that $v = \lambda u$ for some $\lambda \in \mathbf{C}$.

Now assume conversely that u, z are solutions of $\langle u, u \rangle = 0$, $\langle u, Ju \rangle + z^2 = 0$ and that $v = \lambda u$ for some $\lambda \in \mathbf{C}$. Then we have automatically that $\langle u, v \rangle = \lambda \langle u, u \rangle = 0$ and $\langle v, v \rangle = \lambda^2 \langle u, u \rangle = 0$, whereas the equations

$$\begin{aligned} 0 &= \langle u, Jv \rangle + z\zeta = \lambda \langle u, Ju \rangle + z\zeta = -\lambda z^2 + z\zeta = z(\zeta - \lambda z), \\ 0 &= \langle v, Jv \rangle + \zeta^2 = \lambda^2 \langle u, Ju \rangle + \zeta^2 = -\lambda^2 z^2 + \zeta^2 = (\zeta + \lambda z)(\zeta - \lambda z) \end{aligned}$$

are equivalent to $\zeta = \lambda z$ or $z = \zeta + \lambda z = 0$. In the second case $z = \zeta = 0$, and therefore the conclusion is that the equations (10.11) hold if and only if $\zeta = \lambda z$.

In a similar way we obtain that if v, ζ are solutions of $\langle v, v \rangle = 0$, $\langle v, Jv \rangle + \zeta^2 = 0$ and $u = \mu v$, then the equations (10.11) hold if and only if $z = \mu \zeta$. For $\mu \neq 0$ this corresponds to the solutions in the previous paragraph with $\lambda = 1/\mu$, whereas for $\mu = 0$ we obtain the missing solutions with $u = 0$, which implies that $z = 0$ in view of $0 = \langle u, u \rangle + z^2 = z^2$. \square

Suppose that $M(c) = M$ with c and M as in Subsection 10.1. Then M is a smooth two-dimensional affine algebraic variety, and its closure \overline{M} in \mathbf{CP}^8 with respect to the ordinary

topology is a projective algebraic variety, cf. Łojasiewicz [26, p. 383]. It follows that the intersection $M_\infty := \overline{M} \cap \mathbf{CP}_\infty^8$ of \overline{M} with the projective space at infinity, the set of the limit points of M at infinity, is an algebraic variety in \mathbf{CP}_∞^8 . It is known that in general $\dim M_\infty = \dim M - 1$, cf. Łojasiewicz [26, p. 388], and therefore M_∞ is an algebraic curve in the projective space at infinity. (Actually, the explicit computations below lead to an independent verification of this, see Proposition 10.7.)

It follows from Lemma 10.3 that $N(c) = M \cup M(0)_\infty$, which implies that $\overline{M} \subset N(c)$, or $M_\infty = \overline{M} \cap M(0)_\infty$. $N(c)$ is not irreducible, because it has the two-dimensional varieties \overline{M} and $M(0)_\infty$, which intersect along the curve M_∞ , as proper components. As we will see below, the curve $M_\infty = M(c)_\infty$ depends on the choice of the constants c_i , and actually the surface $M(0)_\infty$ is equal to the union of the curves $M(c)_\infty$ for the various c 's such that $\dim M(c) = 2$.

The fact that $M(0)_\infty$ is higher-dimensional than $M(c)_\infty$ is surprising, because for generic polynomials g_i the codimension of $M(0)_\infty$ in the projective space at infinity is equal to the number of the equations. Because the codimension of $M(c)_\infty \subset M(0)_\infty$ cannot be larger, it follows that, for generic g_i , $M(c)_\infty$ is equal to the union of some of the components of $M(0)_\infty$. Compared to this, our set of polynomials g_i is quite degenerate. We still have to determine, in the case that $\dim M(c) = 2 < \dim M(0)$, which curve in the projective surface $M(0)_\infty$ is equal to the limit curve $M(c)_\infty$ of M at infinity.

Remark 10.4 At the subset $M(0)$, both vector fields ξ and η are equal to zero. Actually, the set where both ξ and η are equal to zero is much larger. One component consists of the $((u, z), (v, \zeta))$ for which the vectors (u, z) and (v, ζ) in \mathbf{C}^4 are linearly dependent, this component is five-dimensional. At this component the values c_i of the functions f_i have the property that the matrices

$$F := \begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix} \quad \text{and} \quad G := \begin{pmatrix} c_4 & c_5 \\ c_5 & c_6 \end{pmatrix} \quad (10.12)$$

both are singular and one is a multiple of the other.

Further components consist, for each $i = 1, 2$ or 3 , of the $((u, z), (v, \zeta))$ such that u and v are multiples of e_i and z and ζ are arbitrary. These components are four-dimensional. At this component the matrices F and $G - J_i F$ are singular.

It follows that the condition that $\det F \neq 0$, which allowed us in Subsection 9.2 to make a reduction to the case that $c_2 = 0$ and $c_1 c_3 \neq 0$, implies that we avoid the variety where both vector fields vanish.

Note that if u, z, v, ζ are real, then the equations $\langle u, u \rangle = 0$ and $\langle v, v \rangle = 0$ imply that $u = 0$ and $v = 0$. Subsequently the equations $\langle u, Ju \rangle + z^2 = 0$ and $\langle v, Jv \rangle + \zeta^2 = 0$ imply that $z = 0$ and $\zeta = 0$. Therefore $M(0)_\infty$ has no real points, which implies that M_∞ has no real points either. This corresponds to the fact that the real points of M form a compact subset of \mathbf{R}^8 . \oslash

Suppose that we are at a point $p = ((u, z), (v, \zeta), \epsilon)$ of $M(0)_\infty$, where $\epsilon = 0$, $\langle u, u \rangle = 0$, $\langle u, Ju \rangle + z^2 = 0$, $(u, z) \neq (0, 0)$, and, for some $\lambda \in \mathbf{C}$, $v = \lambda u$ and $\zeta = \lambda z$. Recall that in

the standard charts we have to put one of the coordinates of u, z, v, ζ identically equal to 1. The case $u = 0, z = 0$ is covered by interchanging the role of (u, z) and (v, ζ) .

At such a point p the equation $\sum_{i=1}^6 \alpha_i dg_i(p) = 0$ for the constants $\alpha_i, 1 \leq i \leq 6$, amounts to the equations

$$\begin{aligned} 0 &= (2\alpha_1 + \lambda \alpha_2) u + (2\alpha_4 + \lambda \alpha_5) Ju, & 0 &= (2\alpha_4 + \lambda \alpha_5) z, \\ 0 &= (\alpha_2 + 2\lambda \alpha_3) u + (\alpha_5 + 2\lambda \alpha_6) Ju, & 0 &= (\alpha_5 + 2\lambda \alpha_6) z, \end{aligned}$$

cf. (10.3)—(10.5). If $u \neq 0$, then Ju is linearly independent of u , because otherwise u would be a nonzero multiple of one of the basis vectors e_i , in contradiction with $\langle u, u \rangle = 0$. In that case the equations are equivalent to the four equations

$$2\alpha_1 + \lambda \alpha_2 = 0, \quad 2\alpha_4 + \lambda \alpha_5 = 0, \quad \alpha_2 + 2\lambda \alpha_3 = 0, \quad \alpha_5 + 2\lambda \alpha_6 = 0. \quad (10.13)$$

On the other hand, if $u = 0$ then $z \neq 0$ and the same conclusion holds. It follows that at all points of M_∞ the rank of the matrix of the dg_i 's is equal to four, instead of the expected five.

The equations (10.13) are equivalent to $\alpha_2 = -2\lambda \alpha_3, \alpha_1 = \lambda^2 \alpha_3$ and $\alpha_5 = -\lambda \alpha_6, \alpha_4 = \lambda^2 \alpha_6$, in which α_3 and α_6 are free. In other words, at the aforementioned points of $M(0)_\infty$ we have that the derivatives at p of $\lambda^2 g_1 - 2\lambda g_2 + g_3$ and $\lambda^2 g_4 - 2\lambda g_5 + g_6$ are equal to zero. This implies that $dg_3(p)$ and $dg_6(p)$ are equal to linear combinations of the $dg_j(p)$ with $j = 1, 2, 4, 5$.

Let B denote the common zeroset in \mathbf{CP}^8 of the g_j with $j = 1, 2, 4, 5$. Because the $dg_j(p)$ are linearly independent, we have that near p the set B is a smooth four-dimensional complex projective subvariety of \mathbf{CP}^8 . The tangent space of B at p is equal to the common null space of the $dg_j(p), j = 1, 2, 3, 4$, which in turn is equal to the common null space of the $dg_i(p), 1 \leq i \leq 6$. Note that in the following lemma one of the coordinates of \hat{p} is kept equal to zero, corresponding to the projective coordinate chart in which we are working

Lemma 10.5 *The tangent space of B at p is equal to the common null space of the $dg_i(p), 1 \leq i \leq 6$. It consists of the vectors $\hat{p} = (\hat{z}, \hat{u}, \hat{v}, \hat{\zeta}, \hat{\epsilon})$, such that $\langle u, \hat{u} \rangle = 0, \langle u, \hat{v} \rangle = 0, \langle Ju, \hat{u} \rangle + z \hat{z} = 0$ and $\langle Ju, \hat{v} \rangle + z \hat{\zeta} = 0$.*

Proof The first equation is equivalent to $dg_1(p) \hat{p} = 0$. Assuming $v = \lambda u$ and the first equation, the second equation is equivalent to $dg_2(p) \hat{p} = 0$. The equation $dg_3(p) \hat{p} = 0$ follows from the combination of $v = \lambda u$, the first and the second equation.

The third equation is equivalent to $dg_4(p) \hat{p} = 0$. Assuming $v = \lambda u$ and the third equation, the fourth equation is equivalent to $dg_5(p) \hat{p} = 0$. The equation $dg_6(p) \hat{p} = 0$ follows from the combination of $v = \lambda u$, the third and the fourth equation. \square

Assume that $z \neq 0$, which means that we can work in the projective coordinate system for which $z \equiv 1$. In this case $\zeta = \lambda z = \lambda$. Therefore, if we define the polynomials g and h by

$$h := \zeta^2 g_1 - 2\zeta g_2 + g_3 = \langle \zeta u - v, \zeta u - v \rangle - (c_1 \zeta^2 - 2c_2 \zeta + c_3) \epsilon^2 \quad \text{and} \quad (10.14)$$

$$k := \zeta^2 g_4 - 2\zeta g_5 + g_6 = \langle \zeta u - v, J(\zeta u - v) \rangle - (c_4 \zeta^2 - 2c_5 \zeta + c_6) \epsilon^2, \quad (10.15)$$

respectively, then $dh(p) = 0$ and $dk(p) = 0$ for all $p \in M(0)_\infty$. Moreover, the set $N(c)$, the common zeroset in \mathbf{CP}^8 of *all* the g_i , $1 \leq i \leq 6$, is equal to the zeroset in B of two functions h and k . Note that $M(0)_\infty \subset N(c) \subset B$. Also recall that $M(0)_\infty = N(c) \cap \mathbf{CP}_\infty^8$, that $M = M(c) = N(c) \setminus \mathbf{CP}_\infty^8$, and that $M_\infty = \overline{M} \cap \mathbf{CP}_\infty^8$ is the set of limit points for $\epsilon \rightarrow 0$ of solutions of (10.10) with $\epsilon \neq 0$.

Because h and k vanish up to second order at $p \in M(0)_\infty$, their second order Taylor expansions at p are canonically defined quadratic forms on $T_p B$, given by

$$h^{(2)}(\hat{p}) = \langle \hat{w}, \hat{w} \rangle - (c_1 \zeta^2 - 2c_2 \zeta + c_3) \hat{\epsilon}^2, \quad (10.16)$$

$$k^{(2)}(\hat{p}) = \langle \hat{w}, J \hat{w} \rangle - (c_4 \zeta^2 - 2c_5 \zeta + c_6) \hat{\epsilon}^2, \quad (10.17)$$

in which $\hat{w} := \zeta \hat{u} + \hat{\zeta} u - \hat{v}$. These formulas are obtained by replacing $\zeta u - v$ and ϵ in (10.14) and (10.15) by their first order approximations \hat{w} and $\hat{\epsilon}$.

The equations $\langle u, u \rangle = 0$ and $\langle u, Ju \rangle + 1 = 0$ imply that

$$\det(u, Ju, u \times Ju) = \langle u \times Ju, u \times Ju \rangle = -\langle u, Ju \rangle^2 = -1, \quad (10.18)$$

and therefore the vectors u , Ju and $u \times Ju$ form a basis of \mathbf{C}^3 . Furthermore $z \equiv 1$ implies that $\hat{z} = 0$, and the equations in Lemma 10.5 for $\hat{p} \in T_p B$ imply that $\langle u, \hat{w} \rangle = 0$ and $\langle Ju, \hat{w} \rangle = 0$, which in turn imply that $\hat{w} = \hat{\delta} u \times Ju$ for some $\hat{\delta} \in \mathbf{C}$. Also note that the condition that $\hat{p} \in T_p M(0)_\infty$ is equivalent to $\hat{w} = 0$ and $\hat{\epsilon} = 0$, or $\hat{\delta} = \hat{\epsilon} = 0$.

Let a , b , δ be functions of z , u , v , ζ , which together with ϵ form a regular system of coordinates for B near p , in such a way that $a = b = \delta = \epsilon = 0$ corresponds to the point p and, near p , the equations $\delta = \epsilon = 0$ define $M(0)_\infty$. Then the tangent vector $\partial/\partial\delta$ at the origin corresponds to a tangent vector \hat{p} of $N(c)$ at p , such that $\hat{\epsilon} = 0$ and \hat{p} is not tangent to $M(0)_\infty$. We can arrange this such that $\hat{w} = u \times Ju$. At $\delta = \epsilon = 0$ the functions h and k and their first order derivatives with respect to δ and ϵ are equal to zero. Their Taylor expansions with respect to δ and ϵ start with quadratic terms of the following special form

$$h^{(2)} = h_1(a, b) \delta^2 - h_2(a, b) \epsilon^2, \quad k^{(2)} = k_1(a, b) \delta^2 - k_2(a, b) \epsilon^2. \quad (10.19)$$

The structure of the common zeroset of h and k in B will now be clarified in the following lemma.

Lemma 10.6 *Let $h = h(a, b, \delta, \epsilon)$ and $k = k(a, b, \delta, \epsilon)$ be two holomorphic functions defined in an open neighborhood of the origin in \mathbf{C}^4 . Assume that their Taylor expansion at $\delta = \epsilon = 0$ with respect to δ and ϵ start with quadratic terms as in (10.19). Write $\Delta(a, b) := h_1(a, b) k_2(a, b) - h_2(a, b) k_1(a, b)$, so that the equation $\Delta = 0$ means that the quadratic forms $h^{(2)}$ and $k^{(2)}$ are proportional. If not both $h_1(0, 0)$ and $h_2(0, 0)$ are equal to zero and not both $k_1(0, 0)$ and $k_2(0, 0)$ are equal to zero, then the origin can only be approached by points in the common zeroset of h and k for which $(\delta, \epsilon) \neq (0, 0)$ if $\Delta(0, 0) = 0$.*

If conversely $\Delta(0, 0) = 0$, $h_1(0, 0) \neq 0$, $h_2(0, 0) \neq 0$ and the derivative at $(0, 0)$ of $\Delta(a, b)$ with respect to (a, b) is not equal to zero, then near $(0, 0)$ the common zeroset of h and k is equal to the union of two smooth complex analytic surfaces which intersect cleanly along the smooth curve through the origin which is determined by the equations $\delta = \epsilon = 0$, $\Delta(a, b) = 0$.

Proof Suppose that $h_1(0, 0) \neq 0$ and $h_2(0, 0) \neq 0$, which conditions are equivalent to the condition that $h^{(2)}$ is a nondegenerate quadratic form in δ and ϵ . (If $k_1(0, 0) \neq 0$ and $k_2(0, 0) \neq 0$, then we can interchange the roles of h and k .) Let $\theta = \theta(a, b)$ be a square root of $h_2(a, b)/h_1(a, b)$ which depends holomorphically on (a, b) in a neighborhood of $(0, 0)$. The Morse lemma with parameters, cf. Hörmander [20, Lemma 3.2.3], yields that there is a holomorphic change of the coordinates (δ, ϵ) to coordinates (x, y) , depending holomorphically on (a, b) , such that, near the origin, $h = h_1(a, b)xy$. We can moreover arrange that in first order approximation at $\delta = \epsilon = 0$ we have that $x = \delta + \theta(a, b)\epsilon$ and $y = \delta - \theta(a, b)\epsilon$.

The Taylor expansion of $x \mapsto k(a, b, x, 0)$ at $x = 0$ now starts with a quadratic term, which implies that we can write $k(a, b, x, 0) = K(a, b, x)x^2$, in which $K(a, b, x)$ is a holomorphic function of (a, b, x) near the origin, and

$$K(a, b, 0) := k_1(a, b)/4 - k_2(a, b)/4\theta(a, b)^2 = (h_2 k_1 - h_1 k_2)/4h_2.$$

For $x \neq 0$ the equation $k(a, b, x) = 0$ is equivalent to the equation $K(a, b, x) = 0$, and we conclude that the point p cannot be a limit point of M when $K(0, 0, 0) \neq 0$, or $\Delta(0, 0) \neq 0$.

Assume conversely that $K(0, 0, 0) = 0$, which means that $k_2/k_1 = h_2/h_1$ at $(0, 0)$, and that the derivative at $(0, 0)$ of $(a, b) \mapsto K(a, b, 0)$ is not equal to zero, which is equivalent to the condition that the derivative at $(0, 0)$ of Δ is not equal to zero. For instance, assume that $\partial K(0, b, 0)/\partial b \neq 0$ when $b = 0$. Then the implicit function theorem yields that there exists a holomorphic function $B(a, x)$ of (a, x) near $(0, 0)$ with $B(0, 0) = 0$, such that, for (a, b, x) near $(0, 0, 0)$ the equation $K(a, b, x) = 0$ is equivalent to $b = B(a, x)$. This describes a smooth complex analytic surface, and we obtain the description of the common zero set of h and k near the origin as in the lemma.

The only case which we have not discussed yet is that $\Delta(0, 0) \neq 0$ but not $h_1(0, 0) \neq 0$ and $h_2(0, 0) \neq 0$ and not $k_1(0, 0) \neq 0$ and $k_2(0, 0) \neq 0$, for instance when $h_1(0, 0) \neq 0$, $h_2(0, 0) = 0$, $k_1(0, 0) = 0$, and $k_2(0, 0) \neq 0$. However, in this case we obtain, for an arbitrarily small positive constant c , that the points (a, b, δ, ϵ) near the origin in the zero set of h satisfy an estimate of the form $|\delta| \leq c|\epsilon|$ and those in the zero set of k satisfy $|\epsilon| \leq c|\delta|$, and the conclusion is that $\delta = \epsilon = 0$ for the points (a, b, δ, ϵ) near the origin in the common zero set of h and k . \square

In our case the coefficients in (10.19) are given by

$$\begin{aligned} h_1(a, b) &= \langle u \times Ju, u \times Ju \rangle = -1, & h_2(a, b) &= c_1 \zeta^2 - 2c_2 \zeta + c_3, \\ k_1(a, b) &= \langle u \times Ju, J(u \times Ju) \rangle, & k_2(a, b) &= c_4 \zeta^2 - 2c_5 \zeta + c_6. \end{aligned} \tag{10.20}$$

Until now we did not really use that $M(c) = M$ with c and M as in Subsection 10.1, but from now on this assumption will be essential. Then $h_2 = c_1 \zeta^2 + c_3$ and $k_2 = c_4 \zeta^2 + c_6$. If $h_2 = 0$ then $k_2 \neq 0$ because of the assumption that $c_1 c_6 \neq c_3 c_4$. It follows from Lemma 10.6 that the points of $M(0)_\infty$ where $h_2 = k_1 = 0$ do not belong to \overline{M} .

Inserting $h_2 = c_1 \zeta^2 + c_3$ and $k_2 = c_4 \zeta^2 + c_6$ in (10.20) we obtain that

$$\Delta := h_2 k_1 - h_1 k_2 = (c_1 k_1 + c_4) \zeta^2 + c_3 k_1 + c_6, \tag{10.21}$$

with $k_1 = \langle u \times Ju, J(u \times Ju) \rangle$. Here the vectors u run over the elliptic curve E given by

$$\langle u, u \rangle = 0, \quad \langle u, Ju \rangle + 1 = 0. \quad (10.22)$$

The $u \in E$ together with the free ζ are parametrizing $M(0)_\infty$. The equation $\Delta = 0$ determines a curve in the (u, ζ) -space. We have $d\Delta = 0$ at a zero of Δ if and only if $\Delta = 0$, $(c_1 k_1 + c_4) \zeta = 0$ and $(c_1 \zeta^2 + c_3) dk_1 = 0$.

If $\zeta \neq 0$, then $c_1 k_1 + c_4 = 0$ and $\Delta = 0$ yields that $c_3 k_1 + c_6 = 0$. This leads to a contradiction with the assumption that $c_1 c_6 \neq c_3 c_4$.

If $\zeta = 0$ then $dk_1 = 0$ because $c_1 \zeta^2 + c_3 = c_3 \neq 0$ and $\Delta = 0$ yields that $c_3 k_1 + c_6 = 0$. The tangent space of E is spanned by the vector $u \times Ju$, on which $dk_1 = 0$ if and only if

$$\begin{aligned} 0 &= \langle (u \times Ju) \times Ju, J(u \times Ju) \rangle + \langle u \times J(u \times Ju), J(u \times Ju) \rangle \\ &= \langle u, Ju \rangle \cdot \langle Ju, J(u \times Ju) \rangle - \langle Ju, Ju \rangle \cdot \langle u, J(u \times Ju) \rangle = -\langle Ju, J(u \times Ju) \rangle, \end{aligned}$$

where in the first equality we used that J is symmetric, in the second that $(u \times Ju) \times Ju = \langle u, Ju \rangle Ju - \langle Ju, Ju \rangle u$, and in the last that $\langle u, Ju \rangle = -1$ and once more that J is symmetric. A straightforward calculation shows that

$$\langle Ju, J(u \times Ju) \rangle = (J_1 - J_2)(J_2 - J_3)(J_3 - J_1) u_1 u_2 u_3,$$

which, in view of the assumption that the J_i are different from each other, is equal to zero if and only if $u_i = 0$ for some $i = 1, 2$ or 3 . Writing the indices of the coordinates of u modulo 3, the condition that $u \in E$ now amounts to $u_{i+1}^2 + u_{i+2}^2 = 0$, $J_{i+1} u_{i+1}^2 + J_{i+2} u_{i+2}^2 + 1 = 0$, or $u_{i+1}^2 = 1/(J_{i+2} - J_{i+1})$, $u_{i+2}^2 = 1/(J_{i+1} - J_{i+2})$. This implies that

$$k_1 = \langle u \times Ju, J(u \times Ju) \rangle = J_i (J_{i+2} - J_{i+1})^2 u_{i+1}^2 u_{i+2}^2 = -J_i,$$

and we obtain a contradiction with the equation $c_3 k_1 + c_6 = 0$, in view of the assumption that $c_6 - c_3 J_i \neq 0$.

Applying Lemma 10.6, we obtain the conclusion that in the domain where $z \neq 0$ the curve M_∞ coincides with the subset of $M(0)_\infty$ determined by the equation $\Delta = 0$. It is smooth and near it \overline{M} is equal to the union of two smooth complex analytic surfaces which intersect cleanly along M_∞ .

If $z = 0$ and $u \neq 0$, then we work in a chart where, for some $i = 1, 2$ or 3 , $u_i \equiv 1$. Then $v_i = \lambda u_i = \lambda$ and it becomes expedient to replace the functions h and k in (10.14) and (10.15) near such a point p by

$$\begin{aligned} h &:= v_i^2 g_1 - 2v_i g_2 + g_3 = \langle w, w \rangle - (c_1 v_i^2 + c_3) \epsilon^2 \quad \text{and} \\ k &:= v_i^2 g_4 - 2v_i g_5 + g_6 = \langle w, Jw \rangle + (v_i z - \zeta)^2 - (c_4 v_i^2 + c_6) \epsilon^2, \end{aligned} \quad (10.23)$$

respectively, where $w := v_i u - v$. The manifold $M(0)_\infty$ near p is now parametrized with the curve of the (u, z) with $u_i = 1$, $\langle u, u \rangle = 0$, $\langle u, Ju \rangle + z^2 = 0$, and the coordinate v_i .

At $z = 0$ we have that $\langle u, u \rangle = 0$ and $\langle u, Ju \rangle = 0$, which imply that $\langle u \times Ju, u \times Ju \rangle = 0$. On the other hand it follows from Lemma 10.5 that $\langle u, \hat{u} \rangle = 0$, $\langle u, \hat{v} \rangle = 0$, $\langle Ju, \hat{u} \rangle = 0$,

$\langle Ju, \widehat{v} \rangle = 0$, whereas \widehat{z} and $\widehat{\zeta}$ are free. It follows that $u, \widehat{u}, \widehat{v}$ and therefore also \widehat{w} are multiples of $u \times Ju$, and we conclude that $\langle \widehat{w}, \widehat{w} \rangle = 0$. If we let the vector $\partial/\partial\delta$ in the paragraph preceding (10.19) correspond this time to the vector \widehat{p} such that $\widehat{z} = 0, \widehat{u} = 0, \widehat{v} = 0, \widehat{\zeta} = 1$ and $\widehat{e} = 0$, then we obtain (10.19) with $h_1 = 0, h_2 = c_1 v_i^2 + c_3, k_1 = 1, k_2 = c_4 v_i^2 + c_6$. It follows from Lemma 10.6 that, even when $k_2 = 0$, the point p can only be approached by M if $h_2 = 0$, which implies that $k_2 \neq 0$ because of the assumption that $c_1 c_6 \neq c_3 c_4$. Moreover, because $c_1 \neq 0$, the derivative of $\Delta = h_2 k_1 = c_1 v_i^2 + c_3$ with respect to v_i is nonzero when $\Delta = 0$. Again we can apply Lemma 10.6 and conclude that the curve M_∞ is smooth at p , and that \overline{M} near p is equal to the union of two smooth complex analytic surfaces which intersect cleanly along M_∞ .

The case that $z = 0$ and $u = 0$ is treated by interchanging the role of the vectors (u, z) and (v, ζ) . In this case we use the assumption that $c_4 - c_1 J_i \neq 0$ for every i . Again we can apply Lemma 10.6 and conclude that the curve M_∞ is smooth at p , and that \overline{M} near p is equal to the union of two smooth complex analytic surfaces which intersect cleanly along M_∞ .

The projection $(u, \zeta) \mapsto u$ exhibits M_∞ as a branched covering over the elliptic curve E , where E is defined by (10.22) and $\zeta \in \mathbf{C} \cup \{\infty\}$ are the solutions of $\Delta = 0$, with Δ as in (10.21). Here $\zeta = \infty$ corresponds to $z = 0$, in which case we interchange the role of the vectors (u, z) and (v, ζ) . The branching occurs when $c_3 k_1 + c_6 = 0$ or $c_1 k_1 + c_4 = 0$, and all these branch points are simple. A straightforward calculation shows that the equations $u \in E$ and $k_1 + c = 0$ are equivalent to

$$u_i^2 = (J_i - c) / (J_{i-1} - J_i) (J_i - J_{i+1}), \quad i \in \mathbf{Z}/3\mathbf{Z}. \quad (10.24)$$

It follows that there are $2 \cdot 2^3 = 16$ branch points, all of which are simple, because $J_i - c \neq 0$ for $c = c_4/c_1$ and for $c = c_6/c_3$.

The Riemann-Hurwitz formula says that if one has an n -fold branched covering from a curve Γ onto a curve C and B is the set of branch points in Γ , then

$$\text{genus}(\Gamma) - 1 = n (\text{genus}(C) - 1) + \sum_{b \in B} \text{order}(b)/2, \quad (10.25)$$

cf. Farkas and Kra, [13, p. 18]. Here the order of the branch point b is equal to m if the first m derivatives of the mapping at b are equal to zero. Because the genus of an elliptic curve is equal to one, it follows that the genus of M_∞ minus one is equal to $16/2 = 8$, or the genus of M_∞ is equal to 9. We have proved:

Proposition 10.7 *Suppose that $M(c) = M$, with c and M as in Subsection 10.1. Then $M_\infty := \overline{M} \cap \mathbf{CP}_\infty^8$ is determined by the condition that the quadratic forms $h^{(2)}$ and $k^{(2)}$ are proportional. M_∞ is a smooth closed algebraic curve in \mathbf{CP}_∞^8 of genus equal to 9. Near M_∞ , the variety \overline{M} is equal to the union of two smooth complex analytic surfaces which intersect cleanly along M_∞ .*

The singularities of \overline{M} can be resolved by considering the bundle G over \mathbf{CP}^8 , of which the fiber G_p at $p \in \mathbf{CP}^8$ consists of the space of all two-dimensional linear subspaces of the tangent space at p of \mathbf{CP}^8 . Let G_M denote the restriction of G to M and let τ_M be the section of G_M which is obtained by assigning to $p \in M$ the tangent space $T_p M$ of M at p , which is regarded as an element of G_p . The projection $\pi : \tau_M \rightarrow M$ is an isomorphism from τ_M onto M . Let \widehat{M} denote the closure of τ_M in the projective variety G . Then \widehat{M} is a closed smooth two-dimensional subvariety of G . Define $\widehat{M}_\infty := \widehat{M} \setminus T M$. The projection $\pi : \widehat{M} \rightarrow \overline{M}$ is an isomorphism from the complement τ_M of \widehat{M}_∞ in \widehat{M} , onto the complement M of M_∞ in \overline{M} . On the other hand \widehat{M}_∞ is a smooth closed curve in \widehat{M} and the projection $\pi : \widehat{M}_\infty \rightarrow M_\infty$ is an unbranched two-fold covering.

The mapping $\pi : \widehat{M} \rightarrow \overline{M}$ is a so-called *normalization* of \overline{M} , a regular mapping from an irreducible normal variety (every smooth variety is normal) onto \overline{M} , which is a birational mapping and finite-to-one over every point of \overline{M} , cf. [35, II.5.2]. Because normalizations are unique up to isomorphisms, one talks about *the* normalization of \overline{M} . Our $\pi : \widehat{M} \rightarrow \overline{M}$ is a simple, explicit one.

The Riemann-Hurwitz formula (10.25) yields that the genus of \widehat{M}_∞ minus one is equal to $2 \cdot (9 - 1) = 16$, or that the genus of \widehat{M}_∞ is equal to 17. In this way we obtain a smooth completion \widehat{M} of M which is obtained by adding a smooth curve of genus 17 at infinity. Here the word “completion” is used in the algebraic sense. Proposition 10.8 below says that it can also be used in the sense that the flows of ξ and η , with complex times, are complete on \widehat{M} in the sense that they define a transitive action on \widehat{M}/Σ of the additive group \mathbf{C}^2 .

Proposition 10.8 *Suppose that $M(c) = M$, with c and M as in Subsection 10.1. Let \widehat{M} be the smooth completion of M described above, the normalization of the projective closure of M , which is obtained by adding to M a smooth curve of genus 17 at infinity. Then the rational vector fields ξ and η on \widehat{M} are everywhere finite and linearly independent. Their respective flows $e^{t\xi}$ and $e^{s\eta}$ with complex times t and s define a transitive action of the additive group \mathbf{C}^2 on \widehat{M} , and for each $p \in \widehat{M}$ the mapping $(t, s) \mapsto e^{t\xi} \circ e^{s\eta}(p)$ defines an isomorphism from the complex torus \mathbf{C}^2/Λ onto \widehat{M} . Here*

$$\Lambda := \{(s, t) \mid e^{t\xi} \circ e^{s\eta}(p) = p\}$$

denotes the period lattice. It does not depend on the choice of p and has a \mathbf{Z} -basis consisting of four elements of $\mathbf{C}^2 \simeq \mathbf{R}^4$ which are linearly independent over \mathbf{R} .

Proof We first investigate the vector fields (9.6)—(9.13) near infinity when $z \neq 0$, where we use projective coordinates with $z \equiv 1$. With ϵ as the last coordinate, this means that we identify the affine coordinates $((\tilde{u}, \tilde{z}), (\tilde{v}, \tilde{\zeta}), 1)$ with $p = ((u, 1), (v, \zeta), \epsilon)$, where

$$u = \tilde{z}^{-1} \tilde{u}, \quad v = \tilde{z}^{-1} \tilde{v}, \quad \zeta = \tilde{z}^{-1} \tilde{\zeta}, \quad \epsilon = \tilde{z}^{-1}, \quad \text{or} \quad (10.26)$$

$$\tilde{u} = \epsilon^{-1} u, \quad \tilde{v} = \epsilon^{-1} v, \quad \tilde{\zeta} = \epsilon^{-1} \zeta, \quad \tilde{z} = \epsilon^{-1}. \quad (10.27)$$

Note that our notation means that we have to put tilde's over all the coordinates in the formulas (9.6)—(9.13) for the vector fields ξ and η in the affine coordinate system.

We will write the point p , at which we consider the vector fields ξ and η , as an analytic function of ϵ and a base point p_0 which varies in the curve at infinity. It follows that we have a convergent power series expansion $p = \sum_{j \geq 0} \epsilon^j p_j$, in which the coefficients p_j for $j \geq 1$ depend analytically on the point p_0 in the curve at infinity. We may also assume that the vector p_1 is not tangent to the curve at infinity, which means that it can be identified with the vector $\widehat{p} = \partial/\partial\delta$ in the paragraph preceding (10.19). Recall also that

$$\zeta \widehat{u} + \widehat{\zeta} u - \widehat{v} = \widehat{w} = \theta u \times J u, \quad (10.28)$$

in which the nonzero factor θ is equal to a square root of $-\zeta^2 - 1$. Note that there the coordinates of the base point p_0 are denoted by $((u, 1), (v, \zeta), 0)$, instead of the $((u_0, 1), (v_0, \zeta_0), 0)$ which we will use here.

With these notations, we have that

$$\xi\epsilon = -\widetilde{z}^{-2} \xi \widetilde{z} = -\epsilon^2 \langle \widetilde{u} \times J \widetilde{u}, J \widetilde{v} \rangle = -\epsilon^{-1} \langle u \times J u, J v \rangle.$$

The constant term in the expression following ϵ^{-1} is equal to zero, because $v_0 = \zeta_0 u_0$. The first order term in its Taylor expansion with respect to ϵ is equal to

$$\langle u_1 \times J u_0, J v_0 \rangle + \langle u_0 \times J u_1, J v_0 \rangle + \langle u_0 \times J u_0, J v_1 \rangle = \langle u_0 \times J u_0, J (v_1 - \zeta_0 u_0) \rangle$$

where we again have used that $v_0 = \zeta_0 u_0$. Using (10.28), we obtain that $\xi\epsilon$ attains the finite value

$$\xi\epsilon = \langle u \times J u, J \widehat{w} \rangle \quad \text{at infinity.} \quad (10.29)$$

Using that $u = \epsilon \widetilde{u}$, we subsequently obtain that

$$\xi u = (\xi\epsilon) \widetilde{u} + \epsilon \xi \widetilde{u} = \epsilon^{-2} (\epsilon (\xi\epsilon) u + u \times J (v - \zeta u)),$$

where we have used the homogeneity of ξ of degree 3 and $z \equiv 1$. The constant term in the expression following ϵ^{-2} is equal to zero, because $v_0 = \zeta_0 u_0$. Using (10.29), we obtain that the first order term in its Taylor expansion with respect to ϵ is equal to

$$\langle u \times J u, J \widehat{w} \rangle u + u_1 \times J (v - \zeta u) + u \times J (v_1 - \zeta_1 u - \zeta u_1) = \langle u \times J u, J \widehat{w} \rangle u - u \times J \widehat{w},$$

where we have dropped all the subscripts 0 in the notation. The inner product of this expression with u is equal to zero. Using that $\langle u, J u \rangle = 1$, we obtain that the inner product with $J u$ is equal to zero as well. Finally the inner product with $u \times J u$ is equal to

$$-\langle u \times J \widehat{w}, u \times J u \rangle = \langle u \times (u \times J u), J \widehat{w} \rangle = -\langle u, J \widehat{w} \rangle$$

because $u \times (u \times J u) = \langle u, J u \rangle u - \langle u, u \rangle J u$, $\langle u, J u \rangle = -1$ and $\langle u, u \rangle = 0$. Now it follows from Lemma 10.5 with $z = 1$, $\widehat{z} = 0$, and $\langle J u, u \rangle = -1$ that

$$\langle J u, \widehat{w} \rangle = \zeta \langle J u, \widehat{u} \rangle + \widehat{\zeta} \langle J u, u \rangle - \langle J u, \widehat{v} \rangle = 0.$$

Because u , Ju and $u \times Ju$ form a basis of \mathbf{C}^3 , the conclusion is that the ϵ^{-1} -term in the expansion of ξu in powers of ϵ is equal to zero as well, or that ξu is finite.

Using that ξ is tangent to the surface \widehat{M} , we have obtained sufficient evidence to conclude that ξ is finite in the complement of at most finitely many points of the curve \widehat{M}_∞ . In combination with the rationality of ξ this implies that ξ is finite on \widehat{M} .

For the vector field η we begin with

$$\eta\epsilon = -\epsilon^{-1} \langle u \times Ju, v \rangle.$$

The constant term in the expression following ϵ^{-1} is equal to zero, because $v_0 = \zeta_0 u_0$. The first order term in its Taylor expansion with respect to ϵ is equal to

$$\langle u_1 \times Ju_0, v_0 \rangle + \langle u_0 \times Ju_1, v_0 \rangle + \langle u_0 \times Ju_0, v_1 \rangle = \langle u_0 \times Ju_0, v_1 - \zeta_0 u_0 \rangle$$

where we again have used that $v_0 = \zeta_0 u_0$. Using (10.28), we obtain that $\eta\epsilon$ attains the finite value

$$\eta\epsilon = \langle u \times Ju, \widehat{w} \rangle \quad \text{at infinity.} \quad (10.30)$$

Note that $\widehat{w} = \theta u \times Ju$ for a nonzero factor θ , and that $\langle u \times Ju, u \times Ju \rangle = -1$, cf. (10.18). Therefore $\eta\epsilon = -\theta \neq 0$ at every point on the curve at infinity where $z \neq 0$.

Using that $u = \epsilon \widetilde{u}$, we subsequently obtain that

$$\eta u = (\eta\epsilon) \widetilde{u} + \epsilon \eta \widetilde{u} = \epsilon^{-2} (\epsilon (\eta\epsilon) u + u \times v),$$

where we have used the homogeneity of η of degree 3 and $z \equiv 1$. The constant term in the expression following ϵ^{-2} is equal to zero, because $v_0 = \zeta_0 u_0$. Using (10.30), we obtain that the first order term in its Taylor expansion with respect to ϵ is equal to

$$\langle u \times Ju, \widehat{w} \rangle u + u_1 \times v + u \times v_1 = \langle u \times Ju, \widehat{w} \rangle u - u \times \widehat{w},$$

where we have dropped all the subscripts 0 in the notation. The inner product of this expression with u is equal to zero. Using that $\langle u, Ju \rangle = 1$, we obtain that the inner product with Ju is equal to zero as well. Finally the inner product with $u \times Ju$ is equal to $-\langle u \times \widehat{w}, u \times Ju \rangle = 0$, because $\widehat{w} = \theta u \times Ju$. Again using that u , Ju and $u \times Ju$ form a basis of \mathbf{C}^3 , we obtain that the ϵ^{-1} -term in the expansion of ηu in powers of ϵ is equal to zero as well, or that ηu is finite. In the same way as for ξ , we conclude that the vector field η is finite on \widehat{M} .

Let S denote the set of points in \widehat{M} where ξ and η are linearly dependent. Proposition 10.2 implies that $S \cap M = \emptyset$, which means that S is contained in the curve \widehat{M}_∞ at infinity. Because ξ and η commute, the set S is invariant under the flow of both vector fields, and it follows that at every point of S both vector fields must be tangent to the curve at infinity. Because $\eta\epsilon \neq 0$ at every point of the curve at infinity where $z \neq 0$, we are left with the points at infinity where $z = 0$.

If $z = 0$, then we have $\langle u, u \rangle = 0$ and $\langle u, Ju \rangle = 0$ and it would follow that $u = 0$ if $u_i = 0$ for some i . Therefore, assuming that $u \neq 0$, we have for every i that $u_i \neq 0$. In

the projective coordinate chart where $u_i \equiv 1$ we have that $\epsilon = \tilde{u}_i^{-1}$. It turns out that then $\eta\epsilon = 0$ at $\epsilon = 0$, which means that η is tangent to the curve at infinity when $z = 0$. For this reason we turn to the computation of $\xi\epsilon$, which is equal to the i -th coordinate of

$$-\epsilon^2 \xi \tilde{u} = -\epsilon^{-1} u \times (z J v - \zeta J u).$$

Note that $v_0 = \lambda u_0$ and $\zeta_0 = \lambda z_0$ for the same factor λ , and $z_0 = 0$, which implies that $\zeta_0 = 0$ as well. Therefore the constant term in the expression after ϵ^{-1} is equal to zero and the first order term in its Taylor expansion with respect to ϵ is equal to

$$(\widehat{z}\lambda - \widehat{\zeta}) u \times J u,$$

where we have dropped all the subscripts 0 in the notation. Because $\lambda = v_i$ when $u_i \equiv 1$, we conclude from (10.23) that the factor $\widehat{z}\lambda - \widehat{\zeta}$ is not equal to zero, where we also use that the equations in Lemma 10.5 with $z = 0$ imply that $\langle \widehat{w}, J \widehat{w} \rangle = 0$ when $\widehat{w} = v_i \widehat{u} + \widehat{v}_i u - \widehat{v}$. Because $u \times J u \neq 0$, it follows that for at least one choice of i we obtain that $\xi\epsilon \neq 0$, which proves that ξ is not tangent to the curve at infinity when $z = 0$.

The case that $z = 0$ and $u = 0$ is treated by interchanging the role of the vectors (u, z) and (v, ζ) . Collecting all results, we have proved that $S = \emptyset$, or that ξ and η are linearly independent at every point of \overline{M} .

Using the branched covering over $U_{\mathbb{C}}$ in Subsection 11.1, one obtains that the complex level surface M is connected (in contrast to the real one), and therefore \widehat{M} is connected as well. The remaining conclusions of the proposition now follow by applying the argument of Arnol'd and Avez [3, Appendix 26] as at the end of Section 7. \square

Remark 10.9 In the complex time coordinates on $\widehat{M} \simeq \mathbb{C}^2/\Lambda$, the vector fields ξ and η are constant (and linearly independent). Proposition 10.8 implies that the rotational motion of Chaplygin's sphere with horizontal moment is *algebraically integrable* according to the definition of Adler and van Moerbeke [2, p. 297]. In view of Subsection 9.2, this result remains true for arbitrary non-vertical moment.

A very different proof of the algebraic integrability can be given by means of Chaplygin's integration of the system in terms of hyperelliptic integrals as described in Subsections 11.2 and 11.3. See Subsection 11.4. \oslash

Remark 10.10 The surface M is invariant under the antipodal mapping $x \mapsto -x$. In projective coordinates near infinity, where we take one of the affine coordinates equal to 1, this mapping is given by $\epsilon \mapsto -\epsilon$, keeping the affine coordinates fixed. The set where $\epsilon = 0$ is the projective space at infinity, which belongs to the fixed point set of the antipodal mapping. The coordinates for $\mathbb{CP}^8/\pm 1$ near infinity are obtained by replacing ϵ by ϵ^2 . The antipodal mapping interchanges the two sheets along M_∞ , and it follows that $\overline{M}/\pm 1$ is a smooth variety. Its curve at infinity, $(\overline{M}/\pm 1) \setminus (M/\pm 1)$, is isomorphic to M_∞ .

The antipodal mapping extends to an involution in \widehat{M} without fixed points, which leaves the vector fields ξ and η invariant. It follows that the projection from \widehat{M} to $\overline{M}/\pm 1$ is a

twofold unbranched covering, which intertwines ξ and η with two vector fields on $\overline{M}/\pm 1$, which we also denote by ξ and η , which at every point are regular and linearly independent. Therefore the complex times of the flows of ξ and η lead to an identification of $\overline{M}/\pm 1$ with a complex torus, on which the vector fields ξ and η are constant.

Because the antipodal mapping belongs to the group Σ in (10.31), we obtain an eight-fold unbranched covering $\pi : \overline{M}/\pm 1 \rightarrow \widehat{M}/\Sigma$ such that the projection from \widehat{M} onto \widehat{M}/Σ in Proposition 10.11 below is equal to the composition of the twofold covering from \widehat{M} onto $\overline{M}/\pm 1$, followed by π . In this way the curve $\widehat{M}_\infty/\Sigma$ of genus two is isomorphic to $M_\infty/(\Sigma/\pm 1)$. \oslash

10.3 A Discrete Symmetry Group

Let Σ denote the group of the 16 transformations S in \mathbf{C}^8 of the form

$$S((u, z), (v, \zeta)) = ((\epsilon_1 R u, \epsilon_2 z), (\epsilon_2 R v, \epsilon_1 \zeta)), \quad (10.31)$$

in which $\epsilon_i = \pm 1$ and $R \in \text{SO}(3)$ is a diagonal matrix, with ± 1 on the diagonal, two of them equal to -1 if $R \neq 1$. A straightforward computation show that every $S \in \Sigma$ leaves the functions f_i in (9.14) invariant, and therefore leaves M invariant as well, for any choice of the constants c_i . Moreover, every $S \in \Sigma$ also leaves both vector fields ξ and η invariant. Each linear transformations S has a natural extension to a projective linear transformation of \mathbf{CP}^8 , which leaves \overline{M} invariant. It also has a natural extension to the bundle G mentioned after Proposition 10.8, and this extension leaves the smooth variety \widehat{M} and the vector fields ξ and η on it invariant.

Proposition 10.11 *Suppose that $M(c) = M$, with c and M as in Subsection 10.1. If $S \in \Sigma$ and $S \neq 1$ then S has no fixed points in \widehat{M} .*

As a consequence, the quotient \widehat{M}/Σ is a smooth complex projective algebraic surface. The projection from \widehat{M} onto \widehat{M}/Σ intertwines the vector fields ξ and η with vector fields on \widehat{M}/Σ which we denote by the same symbols. The vector fields ξ and η on \widehat{M}/Σ are regular and linearly independent at every point, and therefore \widehat{M}/Σ is isomorphic to a complex torus as well.

Under the projection from \widehat{M} onto \widehat{M}/Σ , the curve \widehat{M}_∞ of genus 17 is mapped onto a smooth curve $\widehat{M}_\infty/\Sigma$ of genus equal to 2.

Proof Let F denote the set of fixed points of S . Because the vector fields ξ and η are invariant under S , F is invariant under the flows of ξ and η with complex times. Because these flows define a transitive action of \mathbf{C}^2 on \widehat{M} , it follows that F is either void or equal to \widehat{M} . Because it is easily verified that M is not contained in F , the conclusion is that S has no fixed points in \widehat{M} .

The restriction to $\Gamma := \widehat{M}_\infty$ of the projection from \widehat{M} onto \widehat{M}/Σ defines a 16-fold unbranched covering map from Γ onto $C := \widehat{M}_\infty/\Sigma$. The Riemann-Hurwitz formula 10.25

therefore yields that $16 (\text{genus}(C) - 1) = \text{genus}(\Gamma) - 1 = 17 - 1 = 16$, which implies that $\text{genus}(C) - 1 = 1$, or the genus of C is equal to 2. \square

Remark 10.12 Because every curve of genus 2 is hyperelliptic, cf. Farkas and Kra [13, Prop. III.7.2], we conclude that by adding a hyperelliptic curve of genus 2 at infinity, the manifold M/Σ can be completed to a complex torus, on which ξ and η are linearly independent and constant vector fields.

The torus \widehat{M}/Σ is isomorphic to the Jacobi variety of the hyperelliptic curve C which appears in Chaplygin's integration by means of hyperelliptic integrals. See Remark 11.7. \odot

Remark 10.13 Let $\widehat{M}_\infty/\Sigma$ denote the hyperelliptic curve of genus 2 which is added to M/Σ at infinity in order to obtain the torus \widehat{M}/Σ as the completion of M/Σ , cf. Remark 10.12. Let $J^{\text{co}} = \text{diag}(J_2 J_3, J_3 J_1, J_1 J_2)$ be the comatrix of J as defined in (9.30). The rational function

$$M_\infty \ni ((u, z), (v, \zeta)) \mapsto -\langle u, J^{\text{co}} u \rangle / \langle u, J u \rangle$$

induces a twofold branched covering from $\widehat{M}_\infty/\Sigma$ onto \mathbf{CP}^1 , which branches over the points $\lambda = J_i$ (corresponding to $u_i = 0$) for $i = 1, 2, 3$, $\lambda = \infty$ (corresponding to $z = 0$), and the two zeros of the polynomial $p(\lambda)$ given by (9.26). For the role of $p(\lambda)$, see also iv) in Subsection 10.1, or (10.33) where the values b_i of the functions h_i are given in terms of the values c_i of the functions f_i by means of (9.41)–(9.44). It follows that $\widehat{M}_\infty/\Sigma$ is isomorphic to the hyperelliptic curve which is defined by the equation

$$\mu^2 = p(\lambda) \prod_{i=1}^3 (J_i - \lambda)$$

between the projective coordinates (λ, μ) in \mathbf{CP}^2 .

Of the six fixed points of the hyperelliptic involution $(\lambda, \mu) \mapsto (\lambda, -\mu)$, the four corresponding to $\lambda = J_1, J_2, J_3, \infty$ do not depend on the values c_i of the functions f_i , whereas the other two, the zeros of $p(\lambda)$, move freely with the c_i , even with the constants of motion T and j of Chaplygin's sphere. This means that the curves $\widehat{M}_\infty/\Sigma$ are non-isomorphic for the generic variation of the constants of motion, and describe a two-dimensional subvariety of the three-dimensional moduli space of curves of genus two. If we are also vary the constants J_i freely, then there is no restriction on the isomorphism class of the curve $\widehat{M}_\infty/\Sigma$.

Remark 11.8 contains an explicit verification that the curve $\widehat{M}_\infty/\Sigma$ is isomorphic to the hyperelliptic curve C introduced in (11.50). \odot

Question 10.14 As observed in Remark 10.12, the torus \widehat{M}/Σ is isomorphic to the Jacobi variety of the hyperelliptic curve C . According to Remark 10.13, C is isomorphic to the curve $\widehat{M}_\infty/\Sigma$ which is added at infinity to the affine algebraic surface M/Σ in order to obtain the toral completion \widehat{M}/Σ . It follows from Matsusaka [27] that \widehat{M}/Σ is isomorphic to the Jacobi variety of the curve $\widehat{M}_\infty/\Sigma$, and that $\widehat{M}_\infty/\Sigma$ is canonically embedded in its

Jacobi variety \widehat{M}/Σ , if and only if the self-intersection number of the curve $\widehat{M}_\infty/\Sigma$ in \widehat{M}/Σ is equal to two. (I owe this reference to Ben Moonen.) Is it possible to verify directly that the self-intersection number of the curve $\widehat{M}_\infty/\Sigma$ in \widehat{M}/Σ is equal to two? \odot

Remark 10.15 In terms of the parametrization of \widehat{M} by means of the complex times of the flows of the vector fields ξ and η , cf. Proposition 10.8, the condition that S commutes with these flows implies that S is a translation. For each $S \in \Sigma$ we have that $S^2 = 1$, cf. (10.31). Therefore, if we provide $\mathbf{C}^2 \simeq \mathbf{R}^4$ with a real basis with respect to which the period lattice Λ is equal to \mathbf{Z}^4 , we obtain that S is equal to a translation over a vector $v \in (\frac{1}{2}\mathbf{Z})^4/\mathbf{Z}^4$. Because there are $2^4 = 16$ such vectors v , it follows that Σ is equal to the group of all translations of order two in the torus \widehat{M} . In other words, the covering of $\widehat{M} \rightarrow \widehat{M}/\Sigma$ of the complex torus \widehat{M}/Σ is obtained by replacing the period lattice Λ_0 of \widehat{M}/Σ by $\Lambda = 2\Lambda_0$.

The group Σ is closely related to the set of theta characteristics, as discussed in Mumford [31, p. 163].

I owe this remark, and the encouragement that Proposition 10.8 and Proposition 10.11 might be the right picture of the projective completion of M , to Frans Oort. \odot

10.4 Jordan Rizov's answer to Question 10.14

The following answer to Question 10.14 has been kindly provided to me by Jordan Rizov.

Let us collect in i) – vi) below the abstract data we shall be working with.

- i) Following Chapter 9, we consider the affine variety $M \subset \mathbf{C}^8$ defined by the equations (9.14), in which f_i , $1 \leq i \leq 6$ are constants. According to Proposition 10.1, M is a nonsingular two-dimensional complex variety.
- ii) Consider the “standard” embedding $\mathbf{C}^8 \subset \mathbf{CP}^8$ and let \overline{M} be the projective closure of M in \mathbf{CP}^8 with respect to the complex topology. Then \overline{M} is also the projective closure of M in the Zariski topology, because M is defined as the zeroset of polynomials. According to Proposition 10.7, $M_\infty := \overline{M} \setminus M$ is a smooth algebraic curve of genus 9 and \overline{M} is singular along M_∞ .
- iii) Consider the normalization $\pi : \widehat{M} \rightarrow \overline{M}$ of \overline{M} as constructed after Proposition 10.7, where \widehat{M} is a nonsingular two-dimensional projective variety. The preimage \widehat{M}_∞ of M_∞ in \widehat{M} is a nonsingular projective curve and

$$\pi : \widehat{M}_\infty \rightarrow M_\infty$$

is an unramified two-fold covering. Hence, by the Riemann-Hurwitz theorem, the genus of \widehat{M}_∞ is 17.

- iv) Proposition 10.8 says that \widehat{M} is a two-dimensional complex torus, and hence an Abelian variety (it is projective).

- v) As described in the beginning of Subsection 10.3, there is a group Σ of order 16 acting on \mathbf{C}^8 such that its action extends to an action on \overline{M} and on \widehat{M} . Furthermore Proposition 10.11 says that Σ acts freely on \widehat{M} . Hence, by [30, Ch. II, §7, Thm. 1], the quotient map

$$\pi_\Sigma : \widehat{M} \rightarrow \widehat{M}/\Sigma$$

is étale (= an unramified covering map). Moreover, because the action is free, Σ acts as a finite group of translations on the Abelian surface \widehat{M} and by [30, Ch. II, §7, Thm. 4] the quotient \widehat{M}/Σ is an Abelian variety.

- vi) Since π_Σ is étale, the nonsingular curve \widehat{M}_∞ is mapped onto a nonsingular complete curve $\widehat{M}_\infty/\Sigma$ of genus 2.

Before going on with any computations, let us simplify the notations a little bit by putting

$$\begin{aligned}\Gamma &:= \widehat{M}_\infty/\Sigma, \\ S &:= \widehat{M}/\Sigma, \\ K_S &:= \text{the canonical class of } S,\end{aligned}$$

i.e. the divisor class of a top degree differential form. The question posed at the end of Question 10.14 is whether one can compute directly the self-intersection of Γ on S . We will do this using the

Adjunction formula *Let Γ be a nonsingular curve of genus g_Γ on a nonsingular surface S with canonical class K_S . Then the following relation holds*

$$2g_\Gamma - 2 = \Gamma \cdot (\Gamma + K_S).$$

Proof The proof and the construction of the intersection pairing on a nonsingular surface, in an “algebraic” way, can be found in [17, Ch. 5, §1], where Proposition 1.5 is the adjunction formula. An “analytic” proof is given in [15, Ch. 4, §1]. \square

Therefore, in order to compute the self-intersection $\Gamma \cdot \Gamma$ one has to enquire a little bit about the canonical class K_S of S . As we already saw, S is an Abelian surface, and the next result gives all we need.

Fact *If A is an Abelian variety of dimension g , then*

$$\Omega_A^g \simeq \mathcal{O}_A,$$

or equivalently, the canonical class K_A of A is trivial.

Proof See for instance [14, Ch. 1, Prop. 1.5] or [30, Ch. 1, (5)], especially (*) on page 4, and [30, Ch. 2, §4, (4) on p. 42]. \square

The computation Applying the adjunction formula to S and Γ with $g_{\Gamma} = 2$ and K_S trivial, one gets

$$\Gamma \cdot \Gamma = 2 \times 2 - 2 = 2,$$

which yields that the self-intersection number of Γ in S is equal to two.

10.5 The system in $\mathbf{C}^6 = \bigwedge^2 \mathbf{C}^4$

Let h_i be the functions on \mathbf{C}^6 given by (9.36)—(9.39). In this subsection we will investigate the level set $L = L(b)$ defined by the equations $h_1 = b_1$, $h_2 = 0$, $h_3 = b_3$, $h_4 = b_4$. We will assume that the constants b_1, b_3, b_4 satisfy the following conditions:

$$b_1 \neq 0, \quad ((\text{trace } J) b_1 + b_4)^2 \neq b_1 b_3 \quad \text{and} \quad (10.32)$$

$$p(J_i) = b_1 J_i^2 - ((\text{trace } J) b_1 + b_4) J_i + b_3 \neq 0 \quad \text{for } i = 1, 2, 3. \quad (10.33)$$

Let U denote the set of $((u, z), (v, \zeta)) \in \mathbf{C}^8$ such that (u, z) and (v, ζ) are linearly independent, U is an open subset of \mathbf{C}^8 . Let f denote the mapping from U to \mathbf{C}^6 defined by the functions f_i in (9.14). Let V be the smooth hypersurface in $\mathbf{C}^6 \setminus \{0\}$ defined by the equation $h_2 = \langle q, r \rangle = 0$, and let $H : V \rightarrow \mathbf{C}^3$ denote the mapping defined by $H_1 = h_1$, $H_2 = h_3$ and $H_3 = (\text{trace } J) h_1 + h_4$, in which the functions h_i are given by (9.36)—(9.39). Let K be the mapping from \mathbf{C}^6 to \mathbf{C}^3 defined by

$$K(c) = (c_1 c_3 - c_2^2, c_4 c_6 - c_5^2, c_1 c_6 + c_3 c_4 - 2c_2 c_5).$$

Then the equations (9.41)—(9.44) mean that $\bigwedge(\mathbf{C}^8) \subset V$ and $H \circ \bigwedge = K \circ f$.

Let $c \in \mathbf{C}^6$ be such that

$$(b_1, b_3, (\text{trace } J) b_1 + b_4) = K(c). \quad (10.34)$$

Write $F = \begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix}$ and $G = \begin{pmatrix} c_4 & c_5 \\ c_5 & c_6 \end{pmatrix}$. Then the assumptions (10.32), (10.33) just mean that the polynomial $p : \lambda \mapsto \det(G - \lambda F)$ given by (9.28) is of second order and has two distinct zeros, none of these equal to one of the J_i 's. In other words, c satisfies the assumptions in Subsection 10.1. Note that these conditions imply that F and G are linearly independent, which in turn imply that $M(c) \subset U$ and that the Jacobi matrix $T_c K$ at the point c of K is surjective. Let $x \in M(c)$, which means that $f(x) = c$. Proposition 10.1 implies that the Jacobi matrix $T_x f$ at the point x of f is surjective. Write $y = \bigwedge(x)$. Then $H(y) = H(\bigwedge(x)) = K(f(x)) = K(c)$. The chain rule implies that $T_y H \circ T_x \bigwedge = T_c K \circ T_x f$, which is surjective, and therefore $T_y H$ is surjective as well. Using also that \bigwedge intertwines the vector fields ξ and η in \mathbf{C}^8 with the vector fields ξ and η in $\mathfrak{e}(3) \simeq \mathbf{C}^6$, as we have seen in Subsection 9.3, we have proved:

Proposition 10.16 *$L = L(b)$ is a smooth two-dimensional affine subvariety of \mathbf{C}^6 . If (10.34) holds then $\bigwedge|_{M(c)}$ defines a twofold unbranched covering from $M(c)$ onto $L(b)$. It intertwines the vector fields ξ and η on $M(c)$ with the vector fields ξ and η in \mathbf{C}^6 . The latter vector fields are tangent to $L(b)$ and linearly independent at every point of $L(b)$. The mapping $\bigwedge|_{M(c)}$ induces a birational isomorphism from $M(c)/\pm 1$ onto $L(b)$, which we will also denote by \bigwedge .*

The statements that $L(b)$ is smooth and ξ and η are linearly independent at every point of $L(b)$ can also be checked directly, but we found the proof which uses the system in \mathbf{C}^8 simpler.

That the inverse of the rational map $\wedge : M(c)/\pm 1 \rightarrow L(b)$ is rational follows from the general fact that if $f : X \rightarrow Y$ is a rational map between irreducible varieties X and Y of the same dimension, $f(X)$ is dense in Y and f is injective over the preimage of a dense subset of Y , then f is a birational isomorphism. Indeed, the homomorphism f^* from the field $\mathbf{C}(Y)$ of rational functions on Y to the field $\mathbf{C}(X)$ of rational functions on X is injective because $f(X)$ is dense in Y . Furthermore the degree of the field extension of $f^*\mathbf{C}(Y)$ by $\mathbf{C}(X)$ is equal to the number of the elements of the generic fiber of f , cf. [35, II.5.2.Thm. 7] (in characteristic zero every field extension is separable). In our case this implies that f^* is surjective. Clearly f has a rational inverse if and only $f^* : \mathbf{C}(Y) \rightarrow \mathbf{C}(X)$ is an isomorphism.

According to Remark 10.10, the manifold $\overline{M(c)}/\pm 1$ is a complex torus to which the vector fields ξ and η on $M(c)/\pm 1$ extend as constant vector fields. Therefore $\overline{M(c)}/\pm 1$ can be viewed as a toral completion of $L(b)$ to which the vector fields ξ and η on $L(b)$ extend as constant vector fields. This completion is obtained by adding a curve to infinity which is isomorphic to the smooth curve $M(c)_\infty$ of genus 9.

Proposition 10.17 *Let $\overline{M(c)}$ and $\overline{L(b)}$ denote the closure of $M(c)$ and $L(b)$ in \mathbf{CP}^8 and \mathbf{CP}^6 , respectively. Assume that (10.34) holds. Then \wedge extends by continuity to a finite morphism from $\overline{M(c)}$ onto $\overline{L(b)}$, which factorizes through a morphism from $\overline{M(c)}/\pm 1$ onto $\overline{L(b)}$, which we also denote by \wedge . $\overline{M(c)}/\pm 1$ is the complex torus of Remark 10.10 and $\wedge : \overline{M(c)}/\pm 1 \rightarrow \overline{L(b)}$ is a normalization of $\overline{L(b)}$.*

The restriction of \wedge to $(\overline{M(c)}/\pm 1) \setminus (M(c)/\pm 1) = \overline{M(c)} \setminus M(c) = M(c)_\infty$ maps $M(c)_\infty$ onto the curve $L(b)_\infty := \overline{L(b)} \setminus L(b)$ of $L(b)$ at infinity. It assigns to the one-dimensional linear subspace

$$\mathbf{C}((u, z), (v, \zeta), 0) \in M(c)_\infty$$

of \mathbf{C}^9 the one-dimensional linear subspace

$$\mathbf{C}(zu, Ju \times u, 0)$$

of \mathbf{C}^7 , where $(zu, Ju \times u, 0)$ has to be replaced by $(\zeta v, Jv \times v, 0)$ when $u = 0$ and $z = 0$. The image $L(b)_\infty$ is a smooth elliptic curve in \mathbf{CP}^6 . The mapping $\wedge : M(c)_\infty \rightarrow L(b)_\infty$ is a twofold branched covering of the curve $M(c)_\infty$ of genus 9 over the elliptic curve $L(b)_\infty$, where the 16 branch points in $M(c)_\infty$ coincide with the branch points mentioned in the text preceding Proposition 10.7.

Proof The mapping $\wedge : \mathbf{C}^8 \rightarrow \mathbf{C}^6$ extends to a homogeneous polynomial mapping $\wedge : \mathbf{C}^9 \rightarrow \mathbf{C}^7$ of degree two by means of the formula

$$\wedge((u, z), (v, \zeta), \epsilon) = (u \times v, zv - \zeta u, \epsilon^2).$$

Near infinity, where $\epsilon = 0$, we can, as in the proof of Proposition 10.8, write $p = ((u, z), (v, \zeta))$ as a convergent power series $p = \sum_{k \geq 0} \epsilon^k p_k$, in which $\mathbf{C}(p_0, 0)$ varies over the curve $M(c)_\infty$ and the coefficients p_k with $k \geq 1$ depend analytically on p_0 . Because (u_0, z_0) and (v_0, ζ_0) are

linearly dependent when $\mathbf{C}(p_0, 0) \in M(c)_\infty$, we have that $\wedge(p_0) = 0$. If the i -th coordinate of

$$\wedge' := D \wedge(p_0) p_1 = (u_1 \times v_0 + u_0 \wedge v_1, z_1 v_0 + z_0 v_1 - \zeta_1 u_0 - \zeta_0 v_1) \quad (10.35)$$

is nonzero, then a division of all the other coordinates of $\wedge(p)$ by $\wedge(p)_i$ yields the coordinates of $\wedge(p)$ in the standard projective coordinates in which the i -th coordinate is kept equal to 1. Because $\wedge(p)_i = a\epsilon$ in which a has a nonzero limit as $\epsilon \rightarrow 0$, we obtain that $\wedge(p(\epsilon), \epsilon)$ converges in these coordinates as $\epsilon \rightarrow 0$. Moreover, its last coordinate $\epsilon^2 / \wedge(p)_i$ converges to zero as $\epsilon \rightarrow 0$, which means that the limit point belongs to the projective space $\mathbf{CP}_\infty^6 = \mathbf{CP}^6 \setminus \mathbf{C}^6$ at infinity.

The proof of Proposition 10.8 yields that the r -component of the vector \wedge' in (10.35) is nonzero whenever $(p_0, 0) \in M(c)_\infty$. When $z_0 \neq 0$, we can work in the projective coordinate system where $z \equiv 1$, hence $z_0 = 1$, $z_1 = 0$, in which case the second component of \wedge' is equal to the vector $r := v_1 - \zeta_1 u_0 - \zeta_0 v_1$. In the proof of Proposition 10.8 we obtained that $r = \theta u_0 \times J u_0$ for some nonzero factor θ . Using that $v_0 = \zeta_0 u_0$, we find that the p -component of \wedge' then is equal to

$$u_0 \times (v_1 - \zeta_0 v_1) = u_0 \times r = \theta u_0 \times (u_0 \times J u_0) = -\theta u_0,$$

because $\langle u_0, u_0 \rangle = 0$ and $\langle u_0, J u_0 \rangle + 1 = 0$.

If $z_0 = 0$ and $u_0 \neq 0$, then we can work in the projective coordinate system where one of the coordinates $u_{0,i}$ of u_0 is identically equal to 1. We have $\zeta_0 = 0$ and $v_0 = v_{0,i} u_0$, and therefore $r = (z_1 v_{0,i} - \zeta_1) u_0$, where in the proof of Proposition 10.8 we obtained that $z_1 v_{0,i} - \zeta_1 \neq 0$. If z_0 and $u_0 = 0$, then we interchange the roles of the vectors (u_0, z_0) and (v_0, ζ_0) .

This concludes the proof that \wedge has a continuous extension \wedge to $\overline{M(c)}$ which maps $M(c)_\infty$ into the projective space $\mathbf{CP}_\infty^6 \simeq \mathbf{CP}^5$ at infinity. Furthermore, on the dense subset of $M(c)_\infty$ where $z \neq 0$ it matches the description in Proposition 10.17, which therefore is valid at every point of $M(c)_\infty$. We also obtain that for $z \equiv 1$ the restriction of \wedge to $M(c)_\infty$ is equal to the composition of the projection onto u , followed by the embedding $u \mapsto \mathbf{C}(u, J u \times u)$, where u runs over the elliptic curve E defined by $\langle u, u \rangle = 0$, $\langle u, J u \rangle + 1 = 0$. This shows that $\wedge(M(c)_\infty)$ is isomorphic to E and that the restriction of \wedge to $M(c)_\infty$ is a twofold branched covering with the branch points as mentioned in the text preceding Proposition 10.7. (It is easy to verify that the points on $M(c)_\infty$ with $z = 0$ are no branch points.)

The continuity of \wedge , together with $\wedge(M(c)) = L(b)$ implies that $L(b) \subset \wedge(\overline{M(c)}) \subset \overline{L(b)}$. On the other hand, because $\overline{M(c)}$ is compact, the continuity of \wedge also implies that $\wedge(\overline{M(c)})$ is a compact, hence closed subset of \mathbf{CP}^6 . We conclude that $\wedge(\overline{M(c)})$ is equal to the closure $\overline{L(b)}$ of $L(b)$ in \mathbf{CP}^6 .

The graph of $\wedge : \overline{M(c)}/\pm 1 \rightarrow \overline{L(b)}$ is equal to the projective closure of the graph of $\wedge : M(c)/\pm 1 \rightarrow L(b)$, where the latter graph is an affine algebraic variety. It follows that the graph of $\wedge : \overline{M(c)}/\pm 1 \rightarrow \overline{L(b)}$ is an algebraic variety, cf. Łojasiewicz [26, p. 383], which implies that $\wedge : \overline{M(c)}/\pm 1 \rightarrow \overline{L(b)}$ is an algebraic morphism. Because it is everywhere finite

and a birational isomorphism from $M(c)$ onto $L(b)$, it is a normalization of $\overline{L(b)}$. Note that $\overline{M(c)}/\pm 1$ is normal, because it is smooth. \square

Proposition 10.18 *We have $L(b)_\infty = L(0)_\infty$ for every choice of the constants b_i in the equations $h_i = b_i$ which define $L(b)$. $L(0)_\infty$ is a smooth elliptic curve in the projective space $\mathbf{CP}_\infty^6 = \mathbf{CP}^6 \setminus \mathbf{C}^6 \simeq \mathbf{CP}^5$ at infinity.*

Proof Let δ be the additional projective coordinate for \mathbf{CP}^6 such that $\delta = 0$ corresponds to the projective space at infinity. (We had $\delta = \epsilon^2$ in the proof of Proposition 10.17.) Then the equations $h_i = b_i$ for $L(b)$ correspond to the homogenized equations $h_i - b_i \delta^2 = 0$. It follows that $L(b)_\infty$ is contained in the subvariety $L(0)_\infty$ of $\mathbf{CP}_\infty^6 \simeq \mathbf{CP}^5$, which is determined by the equations $\delta = 0$ and $h_i = 0$.

It follows from (9.36), (9.38) and (9.39) that the equations $h_1 = 0$, $h_3 = 0$, $h_4 = 0$ are three independent linear equations for q_1^2 , q_2^2 , q_3^2 , which have the solutions

$$q_i^2 = r_{i+1}^2 / (J_i - J_{i-1}) + r_{i-1}^2 / (J_i - J_{i+1}). \quad (10.36)$$

Here the index i is counted modulo 3 (cyclic notation). This determines the q_i up to their signs in terms of r . Also note that $r = 0$ implies that $q = 0$. because $(q, r) = (0, 0)$ is excluded for the projective space at infinity, we have always that $q \neq 0$.

Let ψ be equal to minus the product of $q_1 r_1 + q_2 r_2 + q_3 r_3$, $-q_1 r_1 + q_2 r_2 + q_3 r_3$, $q_1 r_1 - q_2 r_2 + q_3 r_3$, and $q_1 r_1 + q_2 r_2 - q_3 r_3$. Then, for given r , there exists a solution q of (10.36) and $h_2 = 0$ if and only if there exists a solution q of (10.36) and $\psi = 0$. Two of the eight sign choices for the coordinates q_i , then lead to two solution q and $-q$ of $\langle q, r \rangle = 0$.

On the other hand

$$\psi = \sum_{i \in \mathbf{Z}/3\mathbf{Z}} q_i^4 r_i^4 - 2 \sum_{i \in \mathbf{Z}/3\mathbf{Z}} q_i^2 q_{i+1}^2 r_i^2 r_{i+1}^2. \quad (10.37)$$

Substituting (10.36) in (10.37), we obtain after a straightforward calculation that

$$\psi = \left[\sum_{i \in \mathbf{Z}/3\mathbf{Z}} (J_i - J_{i+1})^2 r_i^2 r_{i+1}^2 \right]^2 / \prod_{i \in \mathbf{Z}/3\mathbf{Z}} (J_i - J_{i+1})^2,$$

and therefore the equation $\psi = 0$ is equivalent to

$$\sum_{i \in \mathbf{Z}/3\mathbf{Z}} (J_i - J_{i+1})^2 r_i^2 r_{i+1}^2 = 0. \quad (10.38)$$

Let F be the curve in \mathbf{CP}^2 defined by (10.38). The mapping $r \mapsto (r_1^2, r_2^2, r_3^2)$ defines a 16-fold branched covering of F over a nondegenerate quadric in \mathbf{CP}^2 , which is isomorphic to \mathbf{CP}^1 . The branching occurs when one of the coordinates of r is equal to zero, in which case another coordinate of r has to be equal to zero as well. Therefore the branching occurs

at the three coordinate axes, where the sheets of the covering are connected to each other. At these points, for instance $r = e_3$, $q_1 = 1/(J_1 - J_2)$, $q_2 = -q_1$, $q_3 = 0$, a straightforward check shows that the derivatives of the functions h_i defined by (9.36), (9.38) and (9.39) are linearly independent. Therefore, although F is singular (has ordinary double points) at the coordinate axes, the curve $M(0)_\infty$ is smooth at the corresponding points (q, r) .

For each $r \in F$ we have two opposite q 's which satisfy (10.36) and $\langle q, r \rangle = 0$. The equations (10.36) have the solution $q = 0$ if and only if

$$r_{i+1}^2 = \frac{J_i - J_{i-1}}{J_{i+1} - J_i} r_{i-1}^2, \quad i \in \mathbf{Z}/3\mathbf{Z}, \quad (10.39)$$

and it is easily verified that these equations imply (10.38). Therefore the projection $(q, r) \mapsto r$ defines a twofold covering from $L(0)_\infty$ onto F , which is branched at the four points in F defined by (10.39). These are smooth points of F . If y is a local analytic coordinate of F near such a point, and we substitute $y = z^2$, then we obtain that the corresponding points $(q, r) \in L(0)_\infty$ can be written as $q = z u(z^2)^2$, $r = r(z^2)$, in which $u(y)$ and $r(y)$ are analytic functions of y and $u(0) \neq 0$. It follows that $L(0)_\infty$ is smooth at $(0, r(0))$, and that this point is a simple branch point for the covering $(q, r) \mapsto r$. We conclude that $L(0)_\infty$ is smooth and connected, and therefore irreducible. Because the curve $L(b)_\infty$ is a component of $L(0)_\infty$, it follows that $L(b)_\infty = L(0)_\infty$. We know already from Proposition 10.17 that $L(b)_\infty$ is an elliptic curve, but the above description can be also be used to verify directly that the curve $L(0)_\infty$ is elliptic. \square

Remark 10.19 At first sight the fact that the curve $L(b)_\infty$ in $\overline{L(b)}$ at infinity does not depend on the values of b_1, b_3, b_4 is quite disturbing. According to Remark 10.10, the quotient of $M(c)_\infty$ by the group $\Sigma/\pm 1$ is isomorphic to $\widehat{M}_\infty/\Sigma$. According to Remark 10.13 its isomorphism class varies in a two-dimensional subvariety of the three-dimensional moduli space of curves of genus two. As a consequence the curves $M(c)_\infty$ in general will not be isomorphic either as we vary the constants of motion. The question is where these moduli appear in the completion of $L(b)$, if the curve at infinity of the projective closure $\overline{L(b)}$ is the same for all b .

The answer is that $L(b)_\infty = L(0)_\infty$ is an ordinary double curve of $\overline{L(b)}$ at all points except the branch points of the twofold covering $\wedge : M(c)_\infty \rightarrow L(0)_\infty$. At these branch points, the variety $\overline{L(b)}$ has worse singularities. (We conjecture that, as in Mumford's appendix to [2], these are ordinary pinch points, where $\overline{L(b)}$ has local analytic equations $x^2 = y z^2$.) According to Proposition 10.17, the branch points in $L(0)_\infty$ of $\wedge : M(c)_\infty \rightarrow L(0)_\infty$ correspond to points $(q, r) \in \mathbf{CP}^5$ for which (10.24) holds with u replaced by q . Here c runs over the two solutions of the equation

$$b_1 c^2 - ((\text{trace } J) b_1 + b_4) c + b_3 = 0,$$

cf. (10.33), and therefore the branch points move as a function of the moduli.

The situation is very much similar to the description of $\overline{F_c}$ in Mumford's appendix to [2], in the text starting with "Thus C is an ordinary double curve of $\overline{F_c}$..." and ending with

“... , hence \tilde{C} has genus 9”, on p. 330 and 331. One difference is that our normalization $\wedge : \overline{M(c)}/\pm 1 \rightarrow \overline{L(b)}$ is a quite simple, concrete one, whereas Mumford’s normalization $\pi : \tilde{F}_c \rightarrow \overline{F}_c$ is abstract.

Another difference is that our normalization is equal to an 8-fold unbranched covering of the Jacobi variety of a hyperelliptic curve of genus 2, a characterization which does not appear in [2]. The symmetry group is the group $\Sigma/\pm 1$, with Σ as in (10.31), which on the (q, r) -space acts by means of the transformations

$$S(q, r) = (\epsilon Rq, Rr),$$

in which $\epsilon = \pm 1$ and R is a diagonal rotation as in (10.31).

If we take $L(b)$ as defined by $h_i = b_i$ with $b_1 \neq 0$, then $S^*h_i = h_i$ when $i \neq 2$, but $h_2 = \langle q, r \rangle$ satisfies $S^*h_2 = -h_2$. If $h_2 \neq 0$, we therefore can only divide out the subgroup of four elements S for which $\epsilon = 1$. According to the Riemann-Hurwitz formula (10.25), the quotient of the curve $M(c)_\infty$ of genus 9 by this group of four elements has genus equal to 3. The possibility of arriving at a curve of genus 2 may therefore be related to the fact that we restricted ourselves to the hypersurface $\langle q, r \rangle = 0$. \oslash

Question 10.20 What happens with the sytem on the surface $h_i = b_i$ when $b_2 \neq 0$? Is it still algebraically integrable? \oslash

10.6 Chaplygin

The themes of Section 10 do not occur in Chaplygin [9]. Because theta functions are defined in terms of complex coordinates, one might argue that the sentence “From (29) we see that u and v can be expressed in terms of theta functions of the two arguments α and τ ” in Chaplygin [9, after (30)] yields implicit evidence that Chaplygin did think of complex variables, as does the sentence “Solving equation (41) gives two real values for the quantity f ” in Chaplygin [9, after (41)]. However, the inequalities between (27) and (28) in Chaplygin [9, §3] indicate that Chaplygin mainly focussed on the real system, whereas he also emphasizes that (41) has real solutions. Completion of the complexified system and tori (real or complex) definitely do not occur at all in Chaplygin [9].

11 Hyperelliptic Integrals

In this section we assume that the moments of inertia I_i are different from each other, that $\rho \neq 0$, that the constants of motion (j, T) are at a nonsingular level. We also assume in this section that the *the moment j of the momentum around the point of contact is nonzero and horizontal*, which means that $\|j\|^2 \neq 0$ and $j_3 = 0$ in (3.9). As shown in Subsection 9.2, the rotational motion with arbitrary nonvertical j can be reduced to this case.

In order to obtain a smooth level surface, we will assume that the kinetic energy T is not equal to any of the critical levels

$$T_{\text{crit},i} := \|j\|^2/2 (I_i + \rho), \quad i = 1, 2, 3, \quad (11.1)$$

of the function T_j on $\text{SO}(3)$, cf. (4.9) with $j_3 = 0$. In order to obtain that the complex level surface is smooth, we will need furthermore that

$$2T\rho \neq \|j\|^2. \quad (11.2)$$

If M has real points, then (11.2) is a consequence of the assumption that $T \neq T_{\text{crit},i}$, because $T < T_{\text{crit},1} = \|j\|^2/2 (I_1 + \rho) < \|j\|^2/2\rho$, cf. (11.1), and therefore $2T\rho < \|j\|^2$. In the case that we allow arbitrary complex values for the parameters I_i , ρ and $\|j\|^2$, we have to add (11.2) to the list of conditions. In other words, we make the same assumptions as in Subsection 10.1.

11.1 The Projection onto the First Vector

Our next goal is to simplify the vector field ξ by means of a suitable substitution of variables in the u -space. We recall that u has the concrete interpretation that $-ru$ is equal to the point of contact on the surface of the sphere in body coordinates, cf. (2.7).

We have the two complex surfaces L and $U_{\mathbf{C}}$ in which L is the surface in $\mathbf{C}^3 \times \mathbf{C}^3$ determined by the equations (3.9) and (3.14) and $U_{\mathbf{C}}$ is the quadric

$$U_{\mathbf{C}} := \{u \in \mathbf{C}^3 \mid \langle u, u \rangle = 1\} \quad (11.3)$$

in \mathbf{C}^3 . The projection $(u, v) \mapsto u$ is a branched covering from L onto $U_{\mathbf{C}}$, branching over the set of $u \in U_{\mathbf{C}}$ for which there exists a solution $v \in \mathbf{C}^3$ of $\langle u, v \rangle = 0$, $\langle v, v \rangle = \|j\|^2$, and (3.14), where the derivatives of $\langle u, v \rangle$, $\langle v, v \rangle$, and $f(u, v)$ with respect to v are linearly dependent. If $X(u) = 0$, then v is a solution of (3.14) if and only if $Y(u, v) = 0$, in which case

$$\partial f(u, v)/\partial v = 2Y(u, v) \partial Y(u, v)/\partial v - X(u) \partial Z(v)/\partial v = 0.$$

Therefore the zeroset of X , which does not contain any real points in view of the remark preceding Lemma 7.1, is contained in the branch locus.

When $u \in U_{\mathbf{C}}$ and $X(u) \neq 0$, then u is a branch point if and only if there exists a solution v of (3.9) and (3.14) such that the vectors u , v and

$$w := X(u)\omega = Y(u, v)Ju + X(u)Jv \quad (11.4)$$

are linearly dependent. Here we have used the formula (3.11) for ω . Assuming that j is not vertical, it follows from (3.8) that u and v are linearly independent, and we obtain that $w = \alpha u + \beta v$ for suitable constants α and β . With the abbreviations $X = X(u)$, $Y = Y(u, v)$, we have

$$\alpha = \langle u, w \rangle = Y \langle u, Ju \rangle + X \langle u, Jv \rangle = Y/\rho,$$

cf. (3.15) and (3.16), and

$$\beta \|j\|^2 = \langle v, w \rangle = Y \langle v, Ju \rangle + X \langle v, Jv \rangle = 2T X,$$

cf. (3.16), (3.14), and (3.17). With the notation

$$\tau := 2T/\|j\|^2, \quad (11.5)$$

the equation $w = \alpha u + \beta v$ takes the form $X (J - \tau) v = Y (1/\rho - J) u$. Because $J^{-1} = I + \rho$, cf. (2.18), we have that $J^{-1} (1 - \rho J) = I$ and it follows that v is equal to a nonzero multiple of $K u$, in which

$$K := [1 - \tau (I + \rho)]^{-1} I. \quad (11.6)$$

Because $\langle u, v \rangle = 0$, we arrive at the conclusion that

$$\langle u, K u \rangle = 0. \quad (11.7)$$

In other words, we have proved that away from $X(u) = 0$ the branch locus is contained in the intersection of the quadric $U_{\mathbf{C}}$ with the quadratic cone defined by (11.7). Because this intersection is irreducible, the conclusion is that the branch locus away from $X(u) = 0$ is equal to the intersection of $U_{\mathbf{C}}$ with the quadratic cone defined by (11.7).

Remark 11.1 The matrix K in (11.6) is a diagonal matrix with eigenvalues equal to

$$K_i = I_i / (1 - \tau (I_i + \rho)) = I_i / (1 - T/T_{\text{crit},i}), \quad (11.8)$$

in which the $T_{\text{crit},i}$ are the critical levels of T_j , cf. (11.5) and (11.1). The assumption that we are on a regular level set is equivalent to the condition that $T \neq T_{\text{crit},i}$ for every $i = 1, 2, 3$. If the level set contains real points, then we have that $T_{\text{crit},3} < T < T_{\text{crit},2} < T_{\text{crit},1}$ or $T_{\text{crit},3} < T_{\text{crit},2} < T < T_{\text{crit},1}$. In the first case K has one negative and two positive eigenvalues and in the second case K has two negative and one positive eigenvalue. Therefore, in both cases the cone defined by (11.7) has nonzero real points. The real cone intersects the real unit sphere U in two closed curves, diametrically opposite to each other.

Because each of the two connected components of the real level surface of (j, T) is a torus, it can only be mapped onto the annular region between these two curves. (This can also be verified by means of explicit calculations, cf. the last part of Remark 11.2.) The images of the quasiperiodic solution curves on the level set, cf. Corollary 8.4, while running around the sphere in this band, run from one of the bounding curves to the other, every time with a contact of order two at the bounding curve. A similar behaviour occurs when j is neither horizontal nor vertical, but in that case the bounding curves are determined by more complicated equations. \oslash

Remark 11.2 If $j_3 = 0$ then, for given $u \in U_{\mathbf{C}}$, the solutions v of the equations (3.9) and (3.14) can be explicitly computed in the following way. Using the equations $\langle v, v \rangle = \|j\|^2$ and (11.5), we can write the kinetic energy equation (3.14) in the form

$$X(u) \langle v, (J - \tau) v \rangle + Y(u, v)^2 = 0, \quad (11.9)$$

which is homogenous (of degree two) in the variable v . The condition that j is horizontal corresponds to the equation

$$\langle u, v \rangle = 0. \quad (11.10)$$

In solving the homogenous equations (11.9) and (11.10) for v with $v_3 \neq 0$ we may put $v_3 = 1$. If $u_2 \neq 0$, then we can solve v_2 in terms of v_1 from (11.10). This leads to a quadratic equation for v_1 of the form

$$a_3(u) v_1^2 + 2b_2(u) v_1 + a_2(u) = 0,$$

in which

$$a_3(u) := X(u) (c_1 u_2^2 + c_2 u_1^2) + (c_1 - c_2)^2 u_1^2 u_2^2, \quad (11.11)$$

$$b_2(u) := [X(u) c_2 + (c_1 - c_2) (c_3 - c_2) u_2^2] u_1 u_3 \quad \text{and} \quad (11.12)$$

$$a_1(u) := X(u) (c_2 u_3^2 + c_3 u_2^2) + (c_2 - c_3)^2 u_2^2 u_3^2. \quad (11.13)$$

Here we have used the abbreviation

$$c_i := 1 / (I_i + \rho) - \tau. \quad (11.14)$$

With these notations, we obtain, using repeatedly that $\langle u, u \rangle = 1$, that

$$v_1 = (-b_2(u) \pm \Delta(u)^{1/2}) / a_3(u), \quad (11.15)$$

in which the discriminant turns out to be given by

$$\Delta_2(u) := b_2(u)^2 - a_3(u) a_1(u) = -c_1 c_2 c_3 u_2^2 X(u) \langle u, K u \rangle / \rho, \quad (11.16)$$

with K as in (11.6).

The solution vector v is determined by taking $v_2 := -(u_1 v_1 + u_3) / u_2$. In order to obtain a vector of length $\|j\|$, we have to replace v by $c v$, in which $c = \pm \|j\| / \|v\|$. We obtain four solutions, consisting of two opposite pairs.

As expected, the discriminant is equal to a multiple of $X(u) \langle u, K u \rangle$. Note that for real u we have real solutions v if and only if $c_1 c_2 c_3 \langle u, K u \rangle \leq 0$. It follows from (11.8), (11.5) and (11.14) that $c_i B_i = I_i / (I_i + \rho) > 0$, and therefore the determinant of $c_1 c_2 c_3 K$ is positive, which implies in view of Remark 11.1 that $c_1 c_2 c_3 K$ has two negative eigenvalues. It follows that the part on the unit sphere where $c_1 c_2 c_3 \langle u, K u \rangle \leq 0$ is the connected annular region, bounded by the two smooth curves determined by the equations $\langle u, u \rangle = 1$, $\langle u, K u \rangle = 0$.

For u in the domain $c_1 c_2 c_3 \langle u, K u \rangle < 0$, the interior of the annulus, we obtain four solutions v , which correspond to four possibilities for the velocity vector $\frac{du}{dt}$. One pair of these velocity vectors correspond to one of the connected components of the level set, and their opposites correspond to the other connected component, cf. the discussion at the end of Section 4. These connected components are mapped to each other by means of a rotation over π which maps j to its opposite, cf. the discussion after Lemma 4.1. \oslash

The intersection of the quadric $\langle u, u \rangle = 1$ with the cone $\langle u, K u \rangle = 0$ is equal to the intersection of the quadric $\langle u, u \rangle = 1$ with the quadric $\langle u, (1 + \beta K) u \rangle = 1$, in which β is

any nonzero constant. If we choose $\beta = \tau$, then we obtain that the branch locus is equal to the intersection with $U_{\mathbf{C}}$ of the quadric defined by the equation

$$\langle u, (1 - \gamma I)^{-1} u \rangle = 1, \quad \text{in which} \quad \gamma := \tau / (1 - \tau \rho). \quad (11.17)$$

The equation $X(u) = 0$ is equivalent to $\langle u, \rho(I + \rho)^{-1} u \rangle = 1$, cf. (3.15) and (2.18). The three quadrics $\langle u, u \rangle = 1$, $X(u) = 0$ and (11.17) therefore belong to the one-parameter family of quadrics $\langle u, (1 - \lambda I)^{-1} u \rangle = 1$, in which the parameter λ takes the values $\lambda = 0$, $\lambda = -1/\rho$ and $\lambda = \gamma$, respectively. The substitution of variables $u = I^{1/2} x$ leads to

$$\langle u, (1 - \lambda I)^{-1} u \rangle = \langle x, I(1 - \lambda I)^{-1} x \rangle = \langle x, (I^{-1} - \lambda)^{-1} x \rangle,$$

and our family of quadrics is turned into the *one-parameter family (pencil) of confocal quadrics*

$$\sum_{i=1}^3 \frac{x_i^2}{a_i - \lambda} = 1, \quad (11.18)$$

in which

$$a_i := 1/I_i, \quad \text{hence} \quad u_i = x_i/a_i^{1/2}. \quad (11.19)$$

11.2 Jacobi's Elliptic Coordinates

Given $x \in \mathbf{C}^3$, the equation (11.18) corresponds to a polynomial equation of degree three for λ , which has three solutions $(\lambda_1, \lambda_2, \lambda_3)$, which are called *Jacobi's elliptic coordinates*, cf. Jacobi [21, 26. Vorlesung]. We briefly recall some of Jacobi's observations.

Let a_i , $1 \leq i \leq n$, be n different numbers, and let $s \in \mathbf{Z}_{\geq 0}$. Then the function $z \mapsto z^s / \prod_k (z - a_k)$ has simple poles at the points $z = a_i$, with residue equal to $a_i^s / \prod_{k|k \neq i} (a_i - a_m)$. It follows that

$$z \mapsto \frac{z^s}{\prod_k (z - a_k)} - \sum_{i=1}^n \frac{a_i^s}{\prod_{k|k \neq i} (a_i - a_m)} \frac{1}{z - a_i}$$

is equal to a polynomial of degree $s - n$ when $s \geq n$ and equal to zero when $s < n$. Therefore, if $s < n$ we have that

$$\frac{z^s}{\prod_k (z - a_k)} = \sum_{i=1}^n \frac{a_i^s}{\prod_{k|k \neq i} (a_i - a_m)} \frac{1}{z - a_i}.$$

If we compare the expansions in powers of $1/z$ for $z \rightarrow \infty$ in the left and right hand side, we obtain that

$$\sum_{i=1}^n \frac{a_i^s}{\prod_{k|k \neq i} (a_i - a_m)} = \begin{cases} 1 & \text{when } s = n - 1, \\ 0 & \text{when } 0 \leq s < n - 1, \end{cases} \quad (11.20)$$

If λ_i , $1 \leq i \leq n$, are arbitrary numbers and we write

$$x_i^2 = y_i = \frac{\prod_k (a_i - \lambda_k)}{\prod_{k|k \neq i} (a_i - a_k)}, \quad (11.21)$$

then an expansion of the numerators in the right hand side of

$$\sum_{i=1}^n \frac{y_i}{a_i - \lambda_l} = \sum_{i=1}^n \frac{\prod_{k|k \neq l} (a_i - \lambda_k)}{\prod_{k|k \neq i} (a_i - a_k)}$$

in powers of a_i yields in combination with (11.20) that

$$\sum_{i=1}^n \frac{y_i}{a_i - \lambda_l} = 1, \quad 1 \leq l \leq n. \quad (11.22)$$

In other words, if the y_i are defined by (11.21), then each of the λ_i 's is a solution of the equation $\sum_i y_i / (a_i - \lambda) = 1$ for λ . Note that (11.21) defines a polynomial mapping $(\lambda_1, \dots, \lambda_n) \mapsto (y_1, \dots, y_n)$ from \mathbf{C}^n to \mathbf{C}^n . It is a branched covering because for every permutation π of $\{1, \dots, n\}$ the vector $(\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)})$ has the same image as $(\lambda_1, \dots, \lambda_n)$. If the x_i satisfy (11.21), then each λ_i is a solution of the equation (11.18). However, (11.21) does not define a mapping $(\lambda_1, \dots, \lambda_n) \mapsto (x_1, \dots, x_n)$, because the mapping $(x_1, \dots, x_n) \mapsto (y_1, \dots, y_n)$ is a 2^n -fold branched covering — for any choice of signs ϵ_i the vector $(\epsilon_1 x_1, \dots, \epsilon_n x_n)$ has the same image as (x_1, \dots, x_n) . The branching occurs at the set where one of the coordinates x_i is equal to zero, which corresponds to the condition that one of the coordinates of λ is equal to a_i .

In our case we have $n = 3$ and (11.19). The sign changes in the x_i correspond to sign changes in the u_i . For the equations of motion in the elliptic coordinates these will not cause too much trouble, because if R is a diagonal matrix with ± 1 's on the diagonal, then the transformation $(u, v) \mapsto (Ru, Rv)$ will leave the vector field invariant when $\det R = 1$ and turns the vector field into its opposite (corresponding with a time reversal) when $\det R = -1$.

We will keep $\lambda_1 = 0$, which implies that the equation $\langle u, u \rangle = 1$ is fulfilled, and regard the remaining two elliptic coordinates (λ_2, λ_3) as coordinates on $U_{\mathbf{C}}$. Apart from the problem that the velocity field for u is multi-valued, it will have singularities at all the branch loci, corresponding to the condition that λ_2 or λ_3 attains any of the five values $-1/\rho$, γ , $1/I_1$, $1/I_2$ or $1/I_3$.

The relation (11.21) induces a relation between tangent vectors X and Λ in the x -space and the λ -space, respectively. If we take the logarithm of the left and right hand side and differentiate, then we obtain that

$$2 \frac{X_i}{x_i} = \sum_k \frac{-\Lambda_k}{a_i - \lambda_k}. \quad (11.23)$$

Squaring this and inserting the formula (11.21) for x_i^2 , we obtain that

$$4X_i^2 = \sum_{k, l|k \neq l} \frac{\prod_{m|m \neq k, m \neq l} (a_i - \lambda_m)}{\prod_{m|m \neq i} (a_i - a_m)} \Lambda_k \Lambda_l + \sum_k \frac{\prod_{m|m \neq k} (a_i - \lambda_m)}{\prod_{m|m \neq i} (a_i - a_m)} \frac{\Lambda_k^2}{a_i - \lambda_k}. \quad (11.24)$$

The sum over i of the first sum in the right hand side, over the k and l with $k \neq l$, vanishes in view of (11.20) with $s < n - 1$. On the other hand

$$\sum_{i=1}^n \frac{\prod_{m|m \neq k} (a_i - \lambda_m)}{\prod_{m|m \neq i} (a_i - a_m)} \cdot \frac{1}{a_i - \lambda_k} = \frac{\prod_{l|l \neq k} (\lambda_l - \lambda_k)}{\prod_l (a_l - \lambda_k)},$$

because as a function of λ_k the left and the right hand side both vanish at infinity and have the same poles and residues. This leads to Jacobi's conclusion, cf. [21, 26. Vorlesung], that

$$4 \sum_{i=1}^n X_i^2 = \sum_k \prod_{l|l \neq k} (\lambda_l - \lambda_k) \frac{\Lambda_k^2}{S(\lambda_k)}, \quad \text{in which} \quad (11.25)$$

$$S(\lambda_k) := \prod_l (a_l - \lambda_k). \quad (11.26)$$

From now on we use that $\lambda_1 = 0$, $\Lambda_1 = 0$, and $n = 3$. Then the quotient of the first sum in the right hand side of (11.24) by a_i vanishes in view of (11.20) with $s \leq n - 3$, where we have used that no terms appear with $k = 1$ or $l = 1$. On the other hand, for every $k \neq 1$ we have that

$$\sum_{i=1}^3 \frac{\prod_{m|m \neq k, m \neq 1} (a_i - \lambda_m)}{\prod_{m|m \neq i} (a_i - a_m)} \cdot \frac{1}{a_i - \lambda_k} = - \frac{\prod_{l|l \neq k, l \neq 1} (\lambda_l - \lambda_k)}{\prod_l (a_l - \lambda_k)},$$

because as a function of λ_k the left and the right hand side both vanish at infinity and have the same poles and residues. With the notation (11.26) we therefore obtain in our case $n = 3$ that

$$4 \sum_{i=1}^3 \frac{X_i^2}{a_i} = (\lambda_2 - \lambda_3) \left(\frac{\Lambda_2^2}{S(\lambda_2)} - \frac{\Lambda_3^2}{S(\lambda_3)} \right). \quad (11.27)$$

For the next equation we will use the index i in a cyclic manner, by taking $i \in \mathbf{Z}/3\mathbf{Z}$. Then (11.23) implies that

$$2 \left(\frac{a_i X_i}{x_i} - \frac{a_{i+1} X_{i+1}}{x_{i+1}} \right) = - \sum_k \left(\frac{a_i}{a_i - \lambda_k} - \frac{a_{i+1}}{a_{i+1} - \lambda_k} \right) \Lambda_k = \sum_k \frac{(a_i - a_{i+1}) \lambda_k \Lambda_k}{(a_i - \lambda_k) (a_{i+1} - \lambda_k)}.$$

On the other hand it follows from (11.21), $\lambda_1 = 0$ and $n = 3$ that

$$\frac{x_i^2 x_{i+1}^2}{a_i a_{i+1}} = \frac{(a_i - \lambda_2) (a_i - \lambda_3) (a_{i+1} - \lambda_2) (a_{i+1} - \lambda_3)}{(a_i - a_{i+1}) (a_i - a_{i+2}) (a_{i+1} - a_{i+2}) (a_{i+1} - a_i)}.$$

It follows that

$$\begin{aligned} & 4 \frac{x_i^2 x_{i+1}^2}{a_i a_{i+1}} \left(\frac{a_i X_i}{x_i} - \frac{a_{i+1} X_{i+1}}{x_{i+1}} \right)^2 \\ &= \frac{(a_i - \lambda_2) (a_i - \lambda_3) (a_{i+1} - \lambda_2) (a_{i+1} - \lambda_3)}{(a_{i+2} - a_i) (a_{i+1} - a_{i+2})} \left(\sum_k \frac{\lambda_k \Lambda_k}{(a_i - \lambda_k) (a_{i+1} - \lambda_k)} \right)^2, \end{aligned}$$

which is equal to $1/(a_{i+2} - a_i)(a_{i+1} - a_{i+2})$ times

$$\frac{(a_i - \lambda_3)(a_{i+1} - \lambda_3)}{(a_i - \lambda_2)(a_{i+1} - \lambda_2)} \lambda_2^2 \Lambda_2^2 + 2\lambda_2 \Lambda_2 \lambda_3 \Lambda_3 + \frac{(a_i - \lambda_2)(a_{i+1} - \lambda_2)}{(a_i - \lambda_3)(a_{i+1} - \lambda_3)} \lambda_3^2 \Lambda_3^2.$$

Now we have

$$\frac{1}{(a_{i+2} - a_i)(a_{i+1} - a_{i+2})} = \frac{a_i - a_{i+1}}{p}, \quad \text{with } p := \prod_{j \in \mathbf{Z}/3\mathbf{Z}} (a_j - a_{j+1}),$$

$$\frac{(a_i - \lambda_3)(a_{i+1} - \lambda_3)}{(a_i - \lambda_2)(a_{i+1} - \lambda_2)} = \frac{S(\lambda_3)}{S(\lambda_2)} \frac{a_{i+2} - \lambda_2}{a_{i+2} - \lambda_3} = \frac{S(\lambda_3)}{S(\lambda_2)} \left(1 + \frac{\lambda_3 - \lambda_2}{a_{i+2} - \lambda_3}\right),$$

the cyclic sum over i of $a_i - a_{i+1}$ is equal to zero, and finally

$$\sum_{i \in \mathbf{Z}/3\mathbf{Z}} \frac{a_i - a_{i+1}}{a_{i+2} - \lambda_3} = \frac{1}{S(\lambda_3)} \sum_{i \in \mathbf{Z}/3\mathbf{Z}} (a_i - a_{i+1})(a_i - \lambda_3)(a_{i+1} - \lambda_3) = -\frac{p}{S(\lambda_3)}.$$

It follows that

$$4 \sum_{i \in \mathbf{Z}/3\mathbf{Z}} \frac{x_i^2 x_{i+1}^2}{a_i a_{i+1}} \left(\frac{a_i X_i}{x_i} - \frac{a_{i+1} X_{i+1}}{x_{i+1}} \right)^2 = (\lambda_2 - \lambda_3) \left(\frac{\lambda_2^2 \Lambda_2^2}{S(\lambda_2)} - \frac{\lambda_3^2 \Lambda_3^2}{S(\lambda_3)} \right). \quad (11.28)$$

11.3 The Motion in Elliptic Coordinates

In order to obtain the time derivative $\Lambda = \dot{\lambda}$ of λ corresponding to the velocity vector $X = \dot{x}$ which in turn corresponds to the time derivative $\dot{u} := u \times \omega$ of u , we start with the observation that the horizontality of j implies that

$$0 = \langle v, u \rangle = \langle I_{\rho, u} \omega, u \rangle = \langle I \omega, u \rangle = \langle \omega, I u \rangle, \quad (11.29)$$

cf. (11.10), (3.11), and (2.16). It follows that

$$\dot{u} \times I u = (u \times \omega) \times I u = \langle u, I u \rangle \omega, \quad (11.30)$$

which equation will be used in order to express ω in terms of \dot{u} .

From (3.7) we obtain that

$$2T = \langle I \omega, \omega \rangle + \rho \langle \dot{u}, \dot{u} \rangle. \quad (11.31)$$

Furthermore,

$$\begin{aligned} \langle I(\dot{u} \times I u), \dot{u} \times I u \rangle &= \sum_{i \in \mathbf{Z}/3\mathbf{Z}} I_i (\dot{u}_{i+1} I_{i+2} u_{i+2} - \dot{u}_{i+2} I_{i+1} u_{i+1})^2 \\ &= \sum_{i \in \mathbf{Z}/3\mathbf{Z}} (I_{i-1} I_{i+1}^2 u_{i+1}^2 + I_{i-2} I_{i-1}^2 u_{i-1}^2) \dot{u}_i^2 - 2I_1 I_2 I_3 \dot{u}_{i+1} u_{i+2} \dot{u}_{i+2} u_{i+1} \\ &= I_1 I_2 I_3 \sum_{i \in \mathbf{Z}/3\mathbf{Z}} \left(\frac{I_{i+1}}{I_i} u_{i+1}^2 + \frac{I_{i-1}}{I_i} u_{i-1}^2 + u_i^2 \right) \dot{u}_i^2 = I_1 I_2 I_3 \langle u, I u \rangle \cdot \langle I^{-1} \dot{u}, \dot{u} \rangle, \end{aligned}$$

where in the third equation we have used that

$$0 = \langle \dot{u}, u \rangle^2 = \sum_{i \in \mathbf{Z}/3\mathbf{Z}} \dot{u}_i^2 u_i^2 + 2\dot{u}_{i+1} u_{i+1} \dot{u}_{i+2} u_{i+2}.$$

It follows that the first term in the equation for the kinetic energy can be written in the form

$$\langle I \omega, \omega \rangle = I_1 I_2 I_3 \langle I^{-1} \dot{u}, \dot{u} \rangle / \langle u, I u \rangle. \quad (11.32)$$

With the substitutions (11.19), we have

$$\langle I^{-1} \dot{u}, \dot{u} \rangle = \sum_{i=1}^3 \dot{x}_i^2 \quad (11.33)$$

and

$$\langle \dot{u}, \dot{u} \rangle = \langle I \dot{x}, \dot{x} \rangle = \sum_{i=1}^3 \dot{x}_i^2 / a_i, \quad (11.34)$$

which by means of (11.25) and (11.27), respectively, can be expressed in terms of the velocities of the elliptic coordinates.

In order to express the denominator

$$\langle u, I u \rangle = \langle x, I^2 x \rangle = \sum_{i=1}^3 x_i^2 / a_i^2 \quad (11.35)$$

in (11.32) in terms of the elliptic coordinates, we observe that

$$1 = \langle u, u \rangle = \sum_{i=1}^3 x_i^2 / a_i \quad (11.36)$$

implies that the third degree polynomial equation, which is obtained from (11.18) by multiplication with $S(\lambda) = \prod_{i=1}^3 (a_i - \lambda)$, has $\lambda = \lambda_1 = 0$ as a solution. The second order equation for the two remaining solutions λ_2, λ_3 takes the form

$$\lambda^2 + \left[\sum_{i=1}^3 (x_i^2 - a_i) \right] \lambda - \sum_{i \in \mathbf{Z}/3\mathbf{Z}} x_i^2 (a_{i+1} + a_{i+2}) + \sum_{h \in \mathbf{Z}/3\mathbf{Z}} a_h a_{h+1} = 0,$$

in which the constant term can be simplified to

$$\sum_{i \in \mathbf{Z}/3\mathbf{Z}} x_i^2 \left[-a_{i+1} - a_{i+2} + \sum_{h \in \mathbf{Z}/3\mathbf{Z}} \frac{a_h a_{h+1}}{a_i} \right] = \sum_{i \in \mathbf{Z}/3\mathbf{Z}} x_i^2 \frac{a_{i+1} a_{i+2}}{a_i} = a_1 a_2 a_3 \sum_{i \in \mathbf{Z}/3\mathbf{Z}} \frac{x_i^2}{a_i^2}.$$

It follows that

$$\lambda_2 + \lambda_3 = \sum_{i=1}^3 (a_i - x_i^2) \quad (11.37)$$

and

$$\lambda_2 \lambda_3 = a_1 a_2 a_3 \sum_{i=1}^3 \frac{x_i^2}{a_i^2}. \quad (11.38)$$

Combining (11.32), (11.33), (11.25) for $X = \dot{x}$, (11.35) and (11.38), we conclude that

$$4\langle I\omega, \omega \rangle = \frac{\lambda_2 - \lambda_3}{\lambda_2 \lambda_3} \left(\frac{\lambda_2 \dot{\lambda}_2^2}{S(\lambda_2)} - \frac{\lambda_3 \dot{\lambda}_3^2}{S(\lambda_3)} \right). \quad (11.39)$$

Combining (11.31) and (11.39) with (11.34) and (11.27) for $X = \dot{x}$, we arrive at

$$8T = \frac{\lambda_2 - \lambda_3}{\lambda_2 \lambda_3} \left(\frac{\lambda_2 \dot{\lambda}_2^2}{S(\lambda_2)} - \frac{\lambda_3 \dot{\lambda}_3^2}{S(\lambda_3)} \right) + \rho (\lambda_2 - \lambda_3) \left(\frac{\dot{\lambda}_2^2}{S(\lambda_2)} - \frac{\dot{\lambda}_3^2}{S(\lambda_2)} \right). \quad (11.40)$$

In order to obtain the second equation for the two unknowns $\dot{\lambda}_2, \dot{\lambda}_3$, we start with the equation

$$\begin{aligned} \|j\|^2 - \rho 2T &= \langle I_{\rho, u} \omega, I_{\rho, u} \omega \rangle - \rho \langle I_{\rho, u} \omega, \omega \rangle \\ &= \langle I\omega + \rho\omega - \rho \langle \omega, u \rangle u, I\omega - \rho \langle \omega, u \rangle u \rangle = \langle I\omega, I\omega \rangle + \rho \langle \omega, I\omega \rangle. \end{aligned}$$

Here we have used (3.9) and (3.7) in the first equation, (2.16) in the second equation, and (11.29) in the third one. The first term in the right hand side is equal to

$$\begin{aligned} \langle I\omega, I\omega \rangle &= \langle u, Iu \rangle^{-2} \cdot \langle I(\dot{u} \times Iu), I(\dot{u} \times Iu) \rangle \\ &= \langle u, Iu \rangle^{-2} \sum_{i \in \mathbf{Z}/3\mathbf{Z}} a_i^{-2} (a_{i+1}^{-1/2} \dot{x}_{i+1} a_{i+2}^{-3/2} x_{i+2} - a_{i+2}^{-1/2} \dot{x}_{i+2} a_{i+1}^{-3/2} x_{i+1}) \\ &= \langle u, Iu \rangle^{-2} (a_1 a_2 a_3)^{-2} \sum_{i \in \mathbf{Z}/3\mathbf{Z}} \frac{x_{i+1}^2 x_{i+2}^2}{a_{i+1} a_{i+2}} \left(\frac{a_{i+1} \dot{x}_{i+1}}{x_{i+1}} - \frac{a_{i+2} \dot{x}_{i+2}}{x_{i+2}} \right)^2, \end{aligned}$$

where we have used (11.30) in the first equation and (11.19) in the second one. Combining this with (11.35), (11.38), and (11.28) for $X = \dot{x}$, we obtain that

$$4\langle I\omega, I\omega \rangle = \frac{\lambda_2 - \lambda_3}{\lambda_2^2 \lambda_3^2} \left(\frac{\lambda_2^2 \dot{\lambda}_2^2}{S(\lambda_2)} - \frac{\lambda_3^2 \dot{\lambda}_3^2}{S(\lambda_3)} \right). \quad (11.41)$$

Also using (11.40), we therefore arrive at

$$4\|j\|^2 - 8\rho T = \frac{\lambda_2 - \lambda_3}{\lambda_2^2 \lambda_3^2} \left(\frac{\lambda_2^2 \dot{\lambda}_2^2}{S(\lambda_2)} - \frac{\lambda_3^2 \dot{\lambda}_3^2}{S(\lambda_3)} \right) + \rho \frac{\lambda_2 - \lambda_3}{\lambda_2 \lambda_3} \left(\frac{\lambda_2 \dot{\lambda}_2^2}{S(\lambda_2)} - \frac{\lambda_3 \dot{\lambda}_3^2}{S(\lambda_3)} \right). \quad (11.42)$$

For the unknowns

$$\xi_2 := (\lambda_2 - \lambda_3) \left(\frac{1}{\lambda_3} + \rho \right) \frac{\dot{\lambda}_2^2}{S(\lambda_2)}, \quad \xi_3 := (\lambda_2 - \lambda_3) \left(\frac{1}{\lambda_2} + \rho \right) \frac{\dot{\lambda}_3^2}{S(\lambda_3)}$$

the equations (11.40) and (11.42) take the form

$$8T = \xi_2 - \xi_3 \quad \text{and} \quad 4\|j\|^2 - 8\rho T = \frac{1}{\lambda_3} \xi_2 - \frac{1}{\lambda_2} \xi_3,$$

respectively. Subtracting λ_2 times the second equation from the first one, we obtain that

$$\left(1 - \frac{\lambda_2}{\lambda_3}\right) (\lambda_2 - \lambda_3) \left(\frac{1}{\lambda_3} + \rho\right) \frac{\dot{\lambda}_2^2}{S(\lambda_2)} = (4\|j\|^2 - 8\rho T) (\gamma - \lambda_2), \quad (11.43)$$

where we have used (11.17) in the second equation.

At this stage we recall the change of the time parametrization defined by

$$\frac{d\tau}{dt} = X(u)^{-1/2},$$

cf. Corollary 8.4 and (3.15). In order to express $X(u)$ in terms of the elliptic coordinates, we recall that

$$\rho X(u) = 1 - \sum_{i=1}^3 \frac{x_i^2}{a_i - \lambda} \quad \text{with} \quad \lambda = -1/\rho.$$

Now

$$S(\lambda) \left(1 - \sum_{i=1}^3 \frac{x_i^2}{a_i - \lambda}\right) = -\lambda (\lambda_2 - \lambda) (\lambda_3 - \lambda), \quad (11.44)$$

because as a polynomial in λ the left and the right hand side have the same zeros $\lambda_1 = 0$, λ_2 and λ_3 and have the same leading coefficient. It follows that

$$X(u) = \rho^{-2} (\lambda_2 + 1/\rho) (\lambda_3 + 1/\rho) / \prod_{i=1}^3 (a_i + 1/\rho). \quad (11.45)$$

Substituting $\dot{\lambda}_2^2 = \left(\frac{d\lambda_2}{d\tau}\right)^2 / X(u)$ and (11.45) in (11.43), we arrive at

$$\left(\frac{d\lambda_2}{d\tau}\right)^2 / P(\lambda_2) = \left(\frac{-c \lambda_3}{\lambda_2 - \lambda_3}\right)^2, \quad (11.46)$$

in which the polynomial P is defined by

$$P(\lambda) := (-1/\rho - \lambda) (\gamma - \lambda) \prod_{i=1}^3 (a_i - \lambda) \quad (11.47)$$

and the constant c is determined by

$$c^2 := (4\|j\|^2 - 8\rho T) / \prod_{i=1}^3 (\rho a_i + 1). \quad (11.48)$$

In a similar way we obtain the equation

$$\left(\frac{d\lambda_3}{d\tau}\right)^2 / P(\lambda_3) = \left(\frac{-c\lambda_2}{\lambda_2 - \lambda_3}\right)^2 \quad (11.49)$$

for $d\lambda_3/d\tau$.

At this point it becomes appropriate to introduce the *hyperelliptic curve* C defined by the polynomial P as the one point completion (compactification) of the affine curve

$$\{(\lambda, z) \in \mathbf{C}^2 \mid z^2 = P(\lambda)\}. \quad (11.50)$$

The genus of the hyperelliptic curve C is equal to g if the degree of the polynomial P is equal to $2g + 1$ or $2g + 2$, cf. Shafarevich [35, Ch. III, §5], or Farkas and Kra [13, III.7.4]. Because in our case the degree of P is equal to five, we have $g = 2$.

Consider the motion on $C \times C$ which is determined by the choice

$$\frac{1}{z_2} \frac{d\lambda_2}{d\tau} = \frac{-c\lambda_3}{\lambda_2 - \lambda_3} \quad \text{and} \quad \frac{1}{z_3} \frac{d\lambda_3}{d\tau} = \frac{-c\lambda_2}{\lambda_2 - \lambda_3} \quad (11.51)$$

of the square roots in (11.46) and (11.49), respectively, where (λ_2, z_2) and (λ_3, z_3) denote the points in C which, under the two-fold branched covering $C \ni (\lambda, z) \mapsto \lambda$, project to λ_2 and λ_3 , respectively.

The equations (11.51) imply that

$$\frac{\lambda_2}{z_2} \frac{d\lambda_2}{d\tau} - \frac{\lambda_3}{z_3} \frac{d\lambda_3}{d\tau} = 0, \quad \frac{1}{z_2} \frac{d\lambda_2}{d\tau} - \frac{1}{z_3} \frac{d\lambda_3}{d\tau} = c, \quad (11.52)$$

or

$$\int_{(\lambda_3(\tau), z_3(\tau))}^{(\lambda_2(\tau), z_2(\tau))} \frac{\lambda}{z} d\lambda = \text{constant}, \quad \int_{(\lambda_3(\tau), z_3(\tau))}^{(\lambda_2(\tau), z_2(\tau))} z^{-1} d\lambda = c\tau + \text{constant}. \quad (11.53)$$

Here the integration is over a curve in C , running from $(\lambda_3(\tau), z_3(\tau))$ to $(\lambda_2(\tau), z_2(\tau))$, and depending smoothly on τ . The primitives of the differential forms $z^{-1} d\lambda$ and $\frac{\lambda}{z} d\lambda$ are the *hyperelliptic integrals* of the hyperelliptic curve C , and for this reason one says that (11.53) implies that *the problem is solved by quadratures in terms of the hyperelliptic integrals* corresponding to the hyperelliptic curve C . The quantities $\lambda_2 - \lambda_3$ and $\lambda_2 \lambda_3$ are given by *Jacobi's theta functions* of the integrals in the left hand sides of (11.53), cf. Shafarevich [35, p. 419], and therefore one also talks about *solving by quadratures in terms of theta functions*.

11.4 The Jacobi Variety of the Hyperelliptic Curve

The equations in (11.53) lead to a beautiful interpretation of the system in terms of the Jacobi variety $\text{Jac}(C)$ of the hyperelliptic curve C . The differential forms

$$\beta_i := \frac{\lambda^i}{z} d\lambda, \quad i = 0, 1, \quad (11.54)$$

extend to holomorphic differential forms on C and actually form a basis of the two-dimensional complex vector space $\mathcal{H}^1(C)$ of all holomorphic differential forms of degree one on C , cf. Shafarevich [35, Ch. III, §5] or Farkas and Kra [13, III.7.5].

Remark 11.3 It is clear that the differential forms

$$\beta'_i := \frac{\lambda_2^i}{z_2} d\lambda_2 - \frac{\lambda_3^i}{z_3} d\lambda_3, \quad i = 0, 1 \quad (11.55)$$

of degree one on $C \times C$ are holomorphic, and closed because of the separation of variables. In this notation, the equations in (11.52) mean that the vector field

$$\xi := \left(\frac{d\lambda_2}{d\tau}, \frac{d\lambda_3}{d\tau} \right) \quad (11.56)$$

on $C \times C$ satisfies

$$\mathbf{i}_\xi \beta'_1 = 0, \quad \mathbf{i}_\xi \beta'_0 = c, \quad (11.57)$$

if $\mathbf{i}_\xi \beta$ denotes the inner product of the differential form β with the vector field ξ . This is a complex version of Proposition 8.3, where we had the differential forms β and γ instead of β'_1, β'_0 , and $\mathbf{i}_\xi \beta = 0$ and $\mathbf{i}_\xi \gamma = -1$. \otimes

For each smooth curve γ in C , we have the complex linear form

$$\mathcal{H}^1(C) \ni \beta \mapsto \int_\gamma \beta \quad (11.58)$$

on $\mathcal{H}^1(C)$ — this defines an element \int_γ of the dual space $\mathcal{H}^1(C)^*$ of $\mathcal{H}^1(C)$. The integral depends only on the homology class $[\gamma] \in H_1(C, \mathbf{Z})$ of γ . If we restrict ourselves to closed loops γ , then this leads to a homomorphism \int from the $H_1(C, \mathbf{Z})$ to $\mathcal{H}^1(C)^*$. The image

$$\Lambda(C) := \int (H_1(C, \mathbf{Z})) \subset \mathcal{H}^1(C)^* \quad (11.59)$$

is an additive subgroup of $\mathcal{H}^1(C)^*$, which is called the *period lattice* of C . For any Riemann surface (complete algebraic curves over \mathbf{C}) of genus g we have that the complex dimension of $\mathcal{H}^1(C)$ is equal to g and therefore the real dimension of $\mathcal{H}^1(C)^*$ is equal to $2g$. Furthermore, the period lattice $\Lambda(C)$ has a \mathbf{Z} -basis which at the same time is an \mathbf{R} -basis of $\mathcal{H}^1(C)^*$, cf. Farkas and Kra [13, III.2.8]. Therefore the quotient space

$$\text{Jac}(C) := \mathcal{H}^1(C)^* / \Lambda(C), \quad (11.60)$$

is compact, a *torus of real dimension equal to $2g$* . Definition (11.60) is the analytic definition of the *Jacobi variety of the curve C* , cf. [13, p. 87] or [31, p. 143], whereas the algebraic definition is formulated in terms of divisors, cf. [35, p. 155] or cite[p. 3.28]TthII.

If we fix $p, q \in C$, then the difference of the homology classes of two curves in C which run from p to q is an element of $H_1(M, \mathbf{Z})$, and it follows that, for each $\beta \in \mathcal{H}^1(C)$, the element

$$\int_p^q := \left(\int_\gamma \right) + \Lambda(C) \in \text{Jac}(C) \quad (11.61)$$

does not depend on the choice of the curve γ from p to q . This defines a smooth mapping $\int : (p, q) \mapsto \int_p^q$ from $C \times C$ to $\text{Jac}(C)$. Note that $\int_q^p = -\int_p^q$ for every $(p, q) \in C \times C$, and $\int_p^p = 0$. If ι denotes the involution $(\lambda, z) \mapsto (\lambda, -z)$ in C , then $\iota^*\beta = -\beta$ for every $\beta \in \mathcal{H}^1(C)$, and it follows that \int maps (p, q) and $(\iota(q), \iota(p))$ to the same element of $\text{Jac}(C)$. Therefore the mapping \int can also be viewed as a mapping to $\text{Jac}(C)$ from the quotient $C \times_{\iota} C$ of $C \times C$ by the involution $\iota' : (p, q) \mapsto (\iota(q), \iota(p))$ in $C \times C$. The diagonal $\{(p, q) \in C \times C \mid p = q\}$ in $C \times C$, which is isomorphic to C , is mapped by the quotient map to a curve D in $C \times_{\iota} C$ which is isomorphic to $C/\iota \simeq \mathbf{P}_1(\mathbf{C})$. Because the diagonal is mapped to the origin, we have that $\int(D) = \{0\}$ as well.

The mapping A is closely related to the *Abel-Jacobi map* $A : C \times C \rightarrow \text{Jac}(C)$, which is defined as follows. Choose a base point $p_0 \in C$. Then $A(p, q)$ is equal to the linear form on $\mathcal{H}^1(C)$ modulo $\Lambda(C)$ which is defined by

$$A(p, q)(\omega) = \int_{p_0}^p \omega + \int_{p_0}^q \omega.$$

Because $A(p, q) = A(q, p)$, A can be viewed as a mapping to $\text{Jac}(C)$ from the *symmetric power* $C^{(2)}$ of C , which is defined as the quotient of $C \times C$ by the involution $(p, q) \mapsto (q, p)$. It is a classical theorem of Abel and Jacobi that for genus two curves C the Abel-Jacobi map is surjective, cf. Farkas and Kra [13, III.6.6]. More precisely, it maps a genus zero curve D_0 in $C^{(2)}$ to a point j_0 in $\text{Jac}(C)$ and is a diffeomorphism from $C^{(2)} \setminus D_0$ onto $\text{Jac}(C) \setminus \{j_0\}$, cf. Farkas and Kra [13, III.11.8 and III.11.11]. This is an example of a *blowing down*, also called a *sigma-process*, cf. Shafarevich [35, Ch. II, §4 and Ch. IV, §3]. Now, if $\iota(p_0) = p_0$, then

$$A(p, q) = \int_{p_0}^q + \int_{p_0}^p = \int_{p_0}^q - \int_{\iota(p_0)}^{\iota(p)} = \int_{p_0}^q - \int_{p_0}^{\iota(p)} = \int_{\iota(p)}^q.$$

The condition that $\iota(p_0) = p_0$ means that p_0 corresponds to one of the five zeros of P or to the point on C at infinity. It follows that the mapping \int also maps a genus zero curve to a point and is a diffeomorphism from the complement of the curve to the complement in

$\text{Jac}(C)$ of the point. Because $\int(D) = \{0\}$, the curve must be equal to D and \int defines a diffeomorphism from $C \times_{\iota} C \setminus D$ onto $\text{Jac}(C) \setminus \{0\}$.

In these terms the equations in (11.53) imply that the tangent map of \int maps the vector field ξ in $C \times C$ to a *constant* vector field on $\text{Jac}(C)$. More precisely, if we identify $\mathcal{H}^1(C)$ with \mathbf{C}^2 by means of the basis β_0, β_1 of (11.54), then (11.53) yields that ξ corresponds to the constant vector field with coordinates $(c, 0)$. We therefore have verified that the rotational motion of Chaplygin's sphere with horizontal moment is *algebraically integrable* according to the definition of Adler and van Moerbeke [2], and in view of Subsection 9.2 this result

remains true if we only assume that the moment is not vertical. This verification is very different from the one via proposition 10.8.

Because every holomorphic vector field on $\text{Jac}(C)$ lifts to a bounded holomorphic vector field on $\mathcal{H}^1(C)^*$, and every bounded holomorphic function on \mathbf{C}^2 is equal to a constant, we have that every holomorphic vector field on $\text{Jac}(C)$ is constant. Therefore our conclusion that ξ is mapped to a constant vector field on $\text{Jac}(C)$ is equivalent to the conclusion that ξ is mapped to a holomorphic vector field on $\text{Jac}(C)$. According to (11.51), the vector field ξ on $C \times C$ is rational with poles along the diagonal. The blowing down of the diagonal by the map \int to a point (the origin) in $\text{Jac}(C)$ apparently has the effect that it regularizes the vector field ξ .

Remark 11.4 If the rotational motion after time parametrization as in Corollary 8.4 is truly quasi-periodic in the sense that the orbit on the (j, T) -level set is dense, then each continuous vector field which commutes with ξ is equal to $a\xi + b\eta$, in which $a = a(j, T)$ and $b = b(j, T)$ are constants which may depend on j and T . Because the levels (j, T) , for which the ξ -orbits are dense in the level set, form a dense subset of the set of all (j, T) , it follows that if a continuous vector field is defined on all the regular level surfaces and also depends continuously on (j, T) , then it is of the form $a\xi + b\eta$ with $a = a(j, T)$ and $b = b(j, T)$ depending continuously on (j, T) . If b_0 and b_1 are arbitrary constants, then the vector field ζ , which in the elliptic coordinates corresponds to

$$i_\zeta \beta'_1 = b_1, \quad i_\zeta \beta'_0 = b_0, \quad (11.62)$$

compare (11.57), or on $\text{Jac}(C)$ corresponds to the constant vector field with coordinates (b_0, b_1) , commutes with ξ and depends continuously on (j, T) . The conclusion is that there is a bijective linear correspondence between the vector fields which commute with ξ and have the aforementioned continuity properties, and the constant vector fields on $\text{Jac}(C)$.

In order to determine the coefficients b_0 and b_1 for which $\zeta = \eta$, where η is as in Proposition 8.3, we start with the time derivative \dot{x} which corresponds to

$$\dot{u} := u \times (I + \rho)\omega = u \times v,$$

where in the second identity we have used (3.11), (2.16), and $u \times u = 0$. It follows that

$$\|j\|^2 = \langle v, v \rangle = \langle u \times v, u \times v \rangle = \langle \dot{u}, \dot{u} \rangle = \sum_{i=1}^3 \frac{\dot{x}_i^2}{a_i}.$$

Here we have used (3.9) in the first equation, $\langle u, v \rangle = -j_3 = 0$ in the second one and $u_i = I_i^{1/2} x_i = a_i^{-1/2} x_i$ in the last one, cf. (11.19). In view of (11.27), we obtain the relation

$$4\|j\|^2 = (\lambda_2 - \lambda_3) \left(\frac{\dot{\lambda}_2^2}{S(\lambda_2)} - \frac{\dot{\lambda}_3^2}{S(\lambda_3)} \right) \quad (11.63)$$

for the corresponding time derivative $\Lambda = \dot{\lambda}$ in elliptic coordinates. We have that $\frac{d\lambda}{d\tau} = X(u)^{1/2} \dot{\lambda}$, with $X(u)$ given by (11.45), satisfies (11.62) if and only if

$$\frac{\lambda_2^i}{z_2} \frac{d\lambda_2}{d\tau} - \frac{\lambda_3^i}{z_3} \frac{d\lambda_3}{d\tau} = b_i, \quad i = 0, 1,$$

or, equivalently,

$$\frac{d\lambda_2}{d\tau}/z_2 = \frac{b_1 - \lambda_3 b_0}{\lambda_2 - \lambda_3}, \quad \frac{d\lambda_3}{d\tau}/z_3 = \frac{b_1 - \lambda_2 b_0}{\lambda_2 - \lambda_3}.$$

Squaring each of the expressions and using (11.50), (11.47), we can bring the equation (11.63) into the form

$$\frac{4\|j\|^2}{\rho^2 \prod_i (a_i + 1/\rho)} = \frac{(\lambda_2 + 1/\rho) (\lambda_2 - \gamma) (b_1 - \lambda_3 b_0)^2 - (\lambda_3 + 1/\rho) (\lambda_3 - \gamma) (b_1 - \lambda_2 b_0)^2}{(\lambda_2 + 1/\rho) (\lambda_2 + 1/\rho) (\lambda_2 - \lambda_3)}.$$

The numerator in the right hand side has to be equal to zero when $\lambda_2 = -1/\rho$, which implies that

$$b_1 = -b_0/\rho. \quad (11.64)$$

Substitution of (11.64) in the previous formula yields that

$$\frac{4\|j\|^2}{\rho^2 \prod_{i=1}^3 (a_i + 1/\rho)} = b_0^2 \frac{(\lambda_2 - \gamma) (\lambda_2 + 1/\rho) - (\lambda_2 - \gamma) (\lambda_3 + 1/\rho)}{\lambda_2 - \lambda_3} = b_0^2 (1/\rho + \gamma),$$

or

$$b_0^2 = 4\|j\|^2/\rho^2 (1/\rho + \gamma) \prod_{i=1}^3 (a_i + 1/\rho), \quad (11.65)$$

a formula similar to (11.48). The formulas (11.65) and (11.64) determine the constant vector field (b_0, b_1) on $\text{Jac}(C)$ corresponding to η , up to its sign. \oslash

Until now we have not paid much attention to the fact that the substitutions which we have used are not bijective but, except for the mapping $\int : C \times_{\iota'} C \rightarrow \text{Jac}(C)$, are branched coverings. Recall that in order to obtain a single-valued vector field ξ , we passed from the manifold L of solutions (u, v) of the equations (3.9) and (3.14) to its two-fold covering M , defined by (9.15).

The projection $(u, v) \mapsto u$ exhibits L as a fourfold branched covering over the quadric

$$U_{\mathbf{C}} := \{u \in \mathbf{C}^3 \mid \langle u, u \rangle = 1\}$$

in \mathbf{C}^3 , with branch locus at $X(u) \langle u, K u \rangle = 0$, cf. Subsection 11.1 and (11.7). At the generic points of $U_{\mathbf{C}}$ we have four possibilities for $\frac{du}{d\tau}$, of the form $\pm\xi'$, $\pm\xi''$, where ξ' and ξ'' become equal at the branch locus.

The branched covering $L \rightarrow U_{\mathbf{C}}$ is less than fourfold over the set D_{∞} of the limit points of the $u \in U_{\mathbf{C}}$ for which at least one of the solutions v of the equations (3.9), (3.14) runs off to infinity. This is the set of $u \in U_{\mathbf{C}}$ for which there exists a nonzero solution v of the homogeneous equations

$$\langle u, v \rangle = 0, \quad \langle v, v \rangle = 0, \quad \langle u, J v \rangle^2 + X(u) \langle v, J v \rangle = 0. \quad (11.66)$$

This exhibits D_{∞} as the resultant set of the three polynomials in v which appear in (11.66), and therefore D_{∞} is a closed algebraic curve in $U_{\mathbf{C}}$.

The set D_∞ contains the set D of the points which correspond to points on the diagonal $\lambda_2 = \lambda_3$ in the elliptic coordinates, because according to (11.51) the vector field in the (λ_2, λ_3) -space is infinite along the diagonal. At the points of D both vectors v and w become infinite, which means that all solutions v of the equations (3.9), (3.14) run off to infinity when u approaches a point in D . This implies that the image of L under the projection $(u, v) \mapsto u$ is contained in $U_{\mathbf{C}} \setminus D$. According to (11.37) and (11.38), the point $u \in U_{\mathbf{C}}$ corresponds to $\lambda_2 = \lambda_3$ in the elliptic coordinates, if and only if

$$\Delta(u) := (\langle I^{-1} u, u \rangle - \text{trace } I^{-1})^2 - 4 \langle I u, u \rangle / \det I = 0. \quad (11.67)$$

Lemma 11.5 *We have that $D_\infty = D$, which implies that the projection $L \rightarrow U_{\mathbf{C}}$ is a fourfold branched covering from L onto $U_{\mathbf{C}} \setminus D$.*

Proof (Sketch.) For $\langle u, u \rangle = 1$ and $u_1^2 + u_2^2 \neq 0$ the nonzero multiples of the $v \in \mathbf{C}^3$ such that

$$v_1 = -u_1 u_3 \pm \sqrt{-1} u_2, \quad v_2 = -u_3 u_2 \mp \sqrt{-1} u_1, \quad v_3 = u_2^2 + u_1^2$$

parametrize the set of nonzero solutions v of the equations $\langle u, v \rangle = 0$ and $\langle v, v \rangle = 0$. Inserting this v in the last equation in (11.66), where we use that $J = (I = \rho)^{-1}$, cf. (2.18), we obtain the polynomial equation

$$\begin{aligned} & [(I_1 - I_3) I_2 u_1^2 + (I_2 - I_3) I_1 u_2^2] [u_1^2 + u_2^2] + (I_2 - I_1) I_3 (u_1^2 - u_2^2) \\ &= 2 \pm \sqrt{-1} (I_2 - I_1) I_3 u_1 u_2 u_3. \end{aligned}$$

Squaring both sides we obtain a polynomial equation of the form

$$\Delta(u) (u_1^2 + u_2^2)^2 = 0,$$

where we have used repeatedly that $\langle u, u \rangle = 1$. A straightforward calculation shows that D_∞ does not contain the points u such that $\langle u, u \rangle = 1$ and $u_1^2 + u_2^2 = 0$, and therefore the equation for D_∞ is equivalent to the equation (11.66) for D . \square

The mapping $u \mapsto y$, $y_i = x_i^2 = a_i u_i^2$ exhibits $U_{\mathbf{C}}$ as an eightfold branched covering of the plane

$$P := \{y \in \mathbf{C}^3 \mid \sum_{i=1}^3 \frac{y_i}{a_i} = 1\},$$

with branch locus at the three coordinate planes, at $y_1 y_2 y_3 = 0$. We arrive at a 64-fold branched covering $M \rightarrow P \setminus D$, where we denoted the image of $D \subset U_{\mathbf{C}}$ in P under the mapping from $U_{\mathbf{C}}$ to P with the same symbol D .

In order to reduce the order of the covering, we will use the group Σ introduced in (10.31). The group Σ leaves the fibers of the 64-fold branched covering $M \rightarrow P \setminus D$ invariant, and

therefore the latter is equal to the composition of the projection $M \rightarrow M/\Sigma$ and a uniquely determined mapping

$$\pi_{M/\Sigma} : M/\Sigma \rightarrow P \setminus D, \quad (11.68)$$

which is a four-fold branched covering.

In the other direction we have the mapping $(\lambda_2, \lambda_3) \mapsto y$ defined by Jacobi's elliptic coordinates, this is a twofold branched covering with branch locus equal to the image of the diagonal $\lambda_2 = \lambda_3$, cf. Subsection 11.2. If we replace the (λ_2, λ_3) -space \mathbf{C}^2 by the symmetric power $\mathbf{C}^{(2)}$, then we actually obtain a birational isomorphism with the inverse of (11.21) given by (11.37), (11.38). (Don Zagier pointed this out to me.)

Let C_{aff} denote the affine (finite) part of the hyperelliptic curve C . Then the projection $(\lambda, z) \mapsto \lambda$ exhibits C_{aff} as a twofold branched covering over \mathbf{C} . This leads to the following commuting diagram of branched coverings:

$$\begin{array}{ccc} C_{\text{aff}} \times C_{\text{aff}} & \xrightarrow{2:1} & C_{\text{aff}} \times_{\iota'} C_{\text{aff}} \\ \downarrow 4:1 & & \downarrow 4:1 \\ \mathbf{C} \times \mathbf{C} & \xrightarrow{2:1} & \mathbf{C}^{(2)} \end{array}$$

Finally we have the mapping $\int : C \times_{\iota'} C \rightarrow \text{Jac}(C)$ which blows down the diagonal D to the origin 0 in $\text{Jac}(C)$ and defines an isomorphism from $(C \times_{\iota'} C) \setminus D$ onto $\text{Jac}(C) \setminus \{0\}$. The image of $C_{\text{aff}} \times_{\iota'} C_{\text{aff}}$ in $\text{Jac}(C)$ is equal to the complement in $\text{Jac}(C)$ of the image of C under the mapping $\int_{\infty} : q \mapsto \int_{\infty}^q$ from C to $\text{Jac}(C)$. (On the real axis, we have that $\infty = -\infty$.) This mapping is an embedding of C into $\text{Jac}(C)$, cf. Farkas and Kra [13, III.6.4]. We denote the image of C in $\text{Jac}(C)$ under this mapping by \int_{∞}^C . Note that $\int_{\infty}^{\infty} = 0$, hence $0 \in \int_{\infty}^C$. It follows that \int defines an isomorphism from $(C_{\text{aff}} \times_{\iota'} C_{\text{aff}}) \setminus D$ onto $\text{Jac}(C) \setminus \int_{\infty}^C$.

The inverse of \int , followed by the projection from $(C_{\text{aff}} \times_{\iota'} C_{\text{aff}}) \setminus D$ to $\mathbf{C}^{(2)} \simeq P$, defines a fourfold branched covering

$$\pi_{\text{Jac}(C)} : \text{Jac}(C) \setminus \int_{\infty}^C \rightarrow P \setminus D. \quad (11.69)$$

In combination with the fourfold branched covering (11.68), this leads to the following identification of M/Σ with $\text{Jac}(C) \setminus \int_{\infty}^C$.

Proposition 11.6 *Let ξ be the velocity vector field on M/Σ and denote by $\xi^{\text{Jac}(C)}$ the constant vector field on $\text{Jac}(C)$ defined by (11.52). Then there is a unique isomorphism ψ from M/Σ onto $\text{Jac}(C) \setminus \int_{\infty}^C$, such that $\pi_{\text{Jac}(C)} \circ \psi = \pi_{M/\Sigma}$, and*

$$T_m \pi_{M/\Sigma} (\xi_m) = T_{\psi(m)} \pi_{\text{Jac}(C)} \left(\xi_{\psi(m)}^{\text{Jac}(C)} \right)$$

for every $m \in M/\Sigma$. In other words, the diagram

$$\begin{array}{ccc} M/\Sigma & \xrightarrow{\psi} & \text{Jac}(C) \setminus \int_{\infty}^C \\ \pi_{M/\Sigma} \searrow & & \swarrow \pi_{\text{Jac}(C)} \\ & P \setminus D & \end{array}$$

commutes and ψ intertwines the vector field ξ on M/Σ with the constant vector field $\xi^{\text{Jac}(C)}$ on $\text{Jac}(C)$. The mapping ψ intertwines the vector field η defined by (9.10) — (9.13) with a constant vector field $\eta^{\text{Jac}(C)}$ on $\text{Jac}(C)$, which up to a sign choice is determined by (11.64), (11.65).

Proof For each $p \in P \setminus D$ there are four (not necessarily distinct) points $m(i) \in M/\Sigma$, $1 \leq i \leq 4$ and $j(k) \in \text{Jac}(C) \setminus \int_{\infty}^C$, $1 \leq k \leq 4$, such that $\pi_{M/\Sigma}(m(i)) = p = \pi_{\text{Jac}(C)}(j(k))$. The relation between the velocity field in the (u, v) -space and in Jacobi's elliptic coordinates, as described in Subsection 11.3, yields that the set $V(p)$ of the $T_{m(i)} \pi_{M/\Sigma}(\xi_{m(i)})$ is equal to the set of the $T_{j(k)} \pi_{\text{Jac}(C)}(\xi_{j(k)}^{\text{Jac}(C)})$. The set P' of all $p \in P \setminus D$ such that $V(p)$ consists of four distinct elements, of the form $\pm \xi'$, $\pm \xi''$, is equal to the complement of a closed curve in $P \setminus D$. The set $M' := \left(\pi_{M/\Sigma}\right)^{-1}(P')$ and $J' := \left(\pi_{\text{Jac}(C)}\right)^{-1}(P')$ is equal to the complement of a closed curve in M/Σ and $\text{Jac}(C) \setminus \int_{\infty}^C$, respectively. The mapping which assigns to $m \in M/\Sigma$ the pair

$$\left(\pi_{M/\Sigma}(m), T_m \pi_{M/\Sigma}(\xi_m)\right)$$

is holomorphic from M/Σ to the tangent bundle TP of P . Its restriction $\Pi_{M'}$ to M' is equal to an analytic diffeomorphism from M' onto a smooth two-dimensional submanifold X'_P of TP . Similarly the mapping which assigns to $j \in \text{Jac}(C) \setminus \int_{\infty}^C$ the pair

$$\left(\pi_{\text{Jac}(C)}(j), T_j \pi_{\text{Jac}(C)}(\xi_j^{\text{Jac}(C)})\right)$$

is holomorphic from $\text{Jac}(C) \setminus \int_{\infty}^C$ to TP , and its restriction $\Pi_{J'}$ to J' is equal to an analytic diffeomorphism from J' onto the same X'_P . It follows that

$$\psi' := (\Pi_{J'})^{-1} \circ \Pi_{M'}$$

is an analytic diffeomorphism from M' onto J' .

The branching properties imply that ψ' has a continuous, hence analytic extension $\psi : M/\Sigma \rightarrow \text{Jac}(C) \setminus \int_{\infty}^C$. Similarly $(\psi')^{-1} : J' \rightarrow M'$ has a continuous, hence analytic extension $\chi : \text{Jac}(C) \setminus \int_{\infty}^C \rightarrow M/\Sigma$. Because $\chi \circ \psi$ is equal to the identity on M' and ψ and χ are continuous, we obtain that $\chi \circ \psi$ is equal to the identity on M/Σ . Similarly we obtain that $\psi \circ \chi$ is equal to the identity on $\text{Jac}(C) \setminus \int_{\infty}^C$. \square

Proposition 11.6 implies that there exists a completion $\widehat{M/\Sigma}$ of M/Σ , obtained by adding a curve at infinity which is isomorphic to the hyperelliptic curve C , such that the mapping ψ in Proposition 11.6 extends to an isomorphism from $\widehat{M/\Sigma}$ onto $\text{Jac}(C)$. The vector fields ξ and η extend to algebraic vector fields on $\widehat{M/\Sigma}$, which commute and are linearly independent at every point of $\widehat{M/\Sigma}$. The word “completion” is meant in the algebraic sense, but it can also be used in the sense that the flows of ξ and η , with complex times, are complete on

\widehat{M}/Σ in the sense that they define a transitive action on \widehat{M}/Σ of the additive group \mathbf{C}^2 . A completion with this property is unique up to isomorphism.

Remark 11.7 In Proposition 10.11 we had obtained such a completion by adding the genus 2 hyperelliptic curve $\widehat{M}_\infty/\Sigma$ at infinity to $M/\Sigma = M/\Sigma$. There the completion is the complex torus \widehat{M}/Σ , in which \widehat{M} is a normalization of the projective closure \overline{M} of M . The isomorphism of Proposition 11.6 leads to an isomorphism between $\text{Jac}(C)$ and \widehat{M}/Σ and an isomorphism between C and the curve $\widehat{M}_\infty/\Sigma$.

The complex torus \widehat{M} is a 16-fold covering of the torus $\widehat{M}/\Sigma \simeq \text{Jac}(C)$, which can also be obtained by replacing the period lattice Λ of $\text{Jac}(C)$ by 2Λ . See Remark 10.15. \circledast

Remark 11.8 The six fixed points of the hyperelliptic involution $(\lambda, z) \mapsto (\lambda, -z)$ of C correspond to $\lambda = 1/I_1, 1/I_2, 1/I_3, -1/\rho, \gamma, \infty$, where γ is given by (11.17), (11.5). Of these, only γ varies when we vary the constants of motion T and j . However, we are working on the assumption that $j_3 = 0$. The transformation of Subsection 9.2 to this case involved that we had to change the values of both T and ρ according to (9.27), and it follows that the isomorphism class of the hyperelliptic curve C varies in a two-dimensional subvariety of the three-dimensional moduli space of curves of genus two, as we vary the constants of motion T and j . If we also vary the moments of inertia I_i freely, then there is no restriction on the isomorphism class of the curve C .

The fractional linear transformation $\lambda \mapsto \lambda/(1 - \rho\lambda)$ maps $J_i = 1/(I_i + \rho)$ to $1/I_i$, ∞ to $-1/\rho$, the zero $\tau = 2T/\|j\|^2$ of $p(\lambda)$ to γ , and the zero $1/\rho$ of $p(\lambda)$ to ∞ . In this description of the zeros of $p(\lambda)$ we have used that $j_3 = 0$. This leads to an explicit verification that the curve $\widehat{M}_\infty/\Sigma$ in Remark 10.13 is isomorphic to C . \circledast

Question 11.9 Can the constant vector fields on $\text{Jac}(C)$ be determined more easily than in Subsection 11.3 by means of calculations at some special points, for instance at points corresponding to one of the fixed points of the hyperelliptic involution ι ? \circledast

Question 11.10 (Richard Cushman) Could the isomorphism in Proposition 11.6 be obtained in a similar way as in Mumford [32, p. 3.57, 3.58] for Neumann's system? This question is suggested by the form (3.14) of the kinetic energy equation. \circledast

11.5 The Translational Motion

In order to obtain the motion of the point of contact $p(t)$, we have to integrate the right hand side of (3.1). Because this vector is horizontal, and the moment of momentum j around the point of contact is assumed to be horizontal as well, it is sufficient to determine the inner product with j and $j \times e_3$. For the latter one we have that

$$\langle (A\omega) \times e_3, j \times e_3 \rangle = \langle A\omega, j \rangle = \langle \omega, A^{-1}j \rangle = \langle \omega, v \rangle = 2T,$$

where we have used (3.8) and (3.12). It follows therefore that *the component of $p(t)$ orthogonal to j grows linearly in time*:

$$\langle p(t), j \times e_3 \rangle = \langle p(0), j \times e_3 \rangle + 2r T t. \quad (11.70)$$

In order to determine the inner product of $\frac{dp}{d\tau}$ with j in terms of Jacobi's elliptic coordinates, we begin with recalling the formula (8.3), in which $\frac{d}{d\tau}\langle p(\tau), j \rangle$ is expressed in terms of the determinant of the vectors u , ω , and $I\omega$. Under a transformation $S \in \Sigma$ as in (10.31), the right hand side of (8.3) gets multiplied by ϵ_2 . Therefore only the square of (8.3) is a single-valued function on M/Σ , whereas (8.3) is single-valued on the unbranched (= unramified) double covering M/Σ_0 of M/Σ , where Σ_0 denotes the group of S in (10.31) such that $\epsilon_2 = 0$. In view of the isomorphism of M/Σ with $\text{Jac}(C)/\int_\infty^C$ in Proposition 11.6, only $\langle \frac{dp}{d\tau}, j \rangle^2$ can be a single valued function of Jacobi's elliptic coordinates.

If C denotes the matrix $(u, \omega, I\omega)$, then

$$(\det C)^2 = \det(C^* \circ C) = \det \begin{pmatrix} \langle u, u \rangle & \langle u, \omega \rangle & \langle u, I\omega \rangle \\ \langle \omega, u \rangle & \langle \omega, \omega \rangle & \langle \omega, I\omega \rangle \\ \langle I\omega, u \rangle & \langle I\omega, \omega \rangle & \langle I\omega, I\omega \rangle \end{pmatrix}.$$

Using that $\langle u, u \rangle = 1$ and $\langle I\omega, u \rangle = 0$, cf. (11.29), we therefore obtain that

$$(\det C)^2 = (\langle \omega, \omega \rangle - \langle u, \omega \rangle^2) \cdot \langle I\omega, I\omega \rangle - \langle \omega, I\omega \rangle^2,$$

in which we can substitute

$$\langle \omega, \omega \rangle - \langle u, \omega \rangle^2 = \langle u \times \omega, u \times \omega \rangle = \langle \dot{u}, \dot{u} \rangle,$$

cf. (3.3). Substituting (11.34) in combination with (11.27) with $X = \dot{x}$, (11.41), and (11.40), we obtain that

$$\begin{aligned} 16(\det C)^2 &= \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2^2 \lambda_3^2} \left\{ \left(\frac{\dot{\lambda}_2^2}{S(\lambda_2)} - \frac{\dot{\lambda}_3^2}{S(\lambda_3)} \right) \left(\frac{\lambda_2^2 \dot{\lambda}_2^2}{S(\lambda_2)} - \frac{\lambda_3^2 \dot{\lambda}_2^2}{S(\lambda_2)} \right) \right. \\ &\quad \left. - \left(\frac{\lambda_2 \dot{\lambda}_2^2}{S(\lambda_2)} - \frac{\lambda_3 \dot{\lambda}_3^2}{S(\lambda_3)} \right)^2 \right\} \\ &= \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2^2 \lambda_3^2} \frac{(-\lambda_3^2 - \lambda_2^2 - 2\lambda_2 \lambda_3) \dot{\lambda}_2^2 \dot{\lambda}_3^2}{S(\lambda_2) S(\lambda_3)} = -\frac{(\lambda_2 - \lambda_3)^4 \dot{\lambda}_2^2 \dot{\lambda}_3^2}{\lambda_2^2 \lambda_3^2 S(\lambda_2) S(\lambda_3)}. \end{aligned}$$

Now we can write (11.43) in the form

$$-\frac{(\lambda_2 - \lambda_3)^2 \dot{\lambda}_2^2}{\lambda_3^2 S(\lambda_2)} = 4 \frac{(\|j\|^2 - 2\rho T)(\gamma - \lambda_2)}{\rho(-1/\rho - \lambda_3)},$$

and we have a similar expression for $\dot{\lambda}_3$, obtained by interchanging the indices 2 and 3. Substituting these, we obtain from (8.3) that

$$\left\langle \frac{dp}{d\tau}, j \right\rangle^2 = r^2 X(u) (\det C)^2 = -\frac{r^2 X(u) (\|j\|^2 - 2\rho T)^2 (\gamma - \lambda_2) (\gamma - \lambda_3)}{\rho^2 (1/\rho + \lambda_3) (1/\rho + \lambda_2)}.$$

In combination with (11.45), this leads to

$$\left(\frac{d\langle p(\tau), j \rangle}{d\tau} \right)^2 = -d^2 (\gamma - \lambda_2) (\gamma - \lambda_3), \quad (11.71)$$

with

$$d^2 := \frac{r^2 (\|j\|^2 - 2\rho T)^2}{\rho^4 \prod_{i=1}^3 (a_i + 1/\rho)}. \quad (11.72)$$

Remark 11.11 It follows from (11.44) with $\lambda = \gamma$ and (11.19) that

$$-(\gamma - \lambda_2) (\gamma - \lambda_3) = \frac{S(\gamma)}{\gamma} \left(1 - \sum_{i=1}^3 \frac{x_i^2}{a_i - \gamma} \right) = \frac{S(\gamma)}{\gamma} (1 - \langle u, (1 - \gamma I)^{-1} u \rangle). \quad (11.73)$$

The right hand side in (11.73) is equal to zero if and only if $u \in U_{\mathbf{C}}$ is in the branch locus for the branched covering $(u, v) \mapsto u$ from L to $U_{\mathbf{C}}$, cf. (11.17).

In fact, we could have concluded this at an earlier stage. The investigation of the branch locus started with the observation that this is the set of points where u , ω and $I\omega$ are linearly dependent, cf. (11.4), and according to (8.3), this is equal to the condition that $d\langle p(t), j \rangle / dt = 0$.

In terms of the description of the real points $u(t)$ on the unit sphere U in Remark 11.1, the instances when $u(t)$ reaches the boundary curves of the annulus in U coincide with the instances that $\langle p(t), j \rangle$ has a turning point. In view of the quasiperiodic motion of the rotational motion, this happens infinitely often. Keeping in mind that the $j \times e_3$ -component of the velocity of $p(t)$ is equal to a positive constant, we obtain that the point of contact $p(t)$ performs a swaying motion in the direction of $j \times e_3$. According to Corollary 8.7, the function $\tau \mapsto \langle p(\tau), j \rangle$ is quasiperiodic if the rotational motion is not periodic and the irrational ratio $\nu_1(\epsilon)/\nu_2(\epsilon)$ mentioned after (5.3) is sufficiently slowly approximated by rational numbers. \otimes

As a function on the Jacobi variety $\text{Jac}(C)$ of the hyperelliptic curve C , the function

$$f := -(\gamma - \lambda_2) (\gamma - \lambda_3)$$

is rational, with poles along \int_{∞}^C , zeros at $\int_{(\gamma, 0)}^C$ and undetermined at the two points of

$$\int_{\infty}^C \cap \int_{(\gamma, 0)}^C = \{0, \int_{\infty}^{(\gamma, 0)}\}. \quad (11.74)$$

Note that $\int_{\infty}^{(\gamma, 0)}$ is a point of order two in $\text{Jac}(C)$, in the sense that $2 \int_{\infty}^{(\gamma, 0)} = 0$.

f has double zeros along $\int_{(\gamma, 0)}^C$, because when λ is near γ , then on the hyperelliptic curve

$$z^2 = P(\lambda) = \prod_{i=1}^3 (a_i - \lambda) (-1/\rho - \lambda) (\gamma - \lambda)$$

the variable z is a local parameter, in terms of which $\gamma - \lambda$ is of order y^2 . Also f has double poles along \int_{∞}^C , because if θ is a local parameter for C at infinity, then $\lambda = \theta^{-2} u$, $z = \theta^{-5} v$, where u and v are units, cf. Shafarevich [35, III.5.5]. Therefore, following a choice of a square root of f along curves in $\text{Jac}(C)$, we find that there exists a rational function g such that $f = g^2$, either on $\text{Jac}(C)$, or on a double covering $\widetilde{\text{Jac}(C)}$, unbranched, of $\text{Jac}(C)$. Ben Moonen told me that g cannot be single-valued on $\text{Jac}(C)$. Actually, if in the choice of the basis of $\Lambda(C)$ corresponding to the closed curves A_1, A_2, B_1, B_2 as in Mumford [32, p. 3.76], with his point a_1 in the role of $(\gamma, 0)$, then g turns into its opposite if we follow A_1 around, and remains unchanged if we follow any of the other curves. Therefore g is a single valued rational function on the unbranched double covering

$$\widetilde{\text{Jac}(C)} = \mathcal{H}^1(C)^*/\Lambda_0(C),$$

where $\Lambda_0(C)$ is the sublattice of index two in $\Lambda(C)$ which is generated by the basis elements corresponding to $2A_1, A_2, B_1, B_2$.

It follows that g is a rational function on $\widetilde{\text{Jac}(C)}$ with simple poles along the preimage of \int_{∞}^C under the double covering $\widetilde{\text{Jac}(C)} \rightarrow \text{Jac}(C)$, simple zeros along the preimage of $\int_{(\gamma, 0)}$, and undetermined at the preimage of (11.74). These properties characterize the function g on $\widetilde{\text{Jac}(C)}$ up to a constant factor. The function g can be identified with a quotient of theta functions as in Mumford [32, p. 3.80, 81], [31, Ch. II].

Question 11.12 We know that

$$\frac{d\langle p(\tau), j \rangle}{d\tau} = r X(u(\tau))^{1/2} \langle A(\tau) \omega(\tau), e_3 \times j \rangle \quad (11.75)$$

is a quasiperiodic function of τ , because the rotational motion is a quasiperiodic function of τ . We now have the additional information that the derivative of $\langle p(\tau), j \rangle$ is given by dg along a straight line path in $\widetilde{\text{Jac}(C)}$, where g is the quotient of theta functions described above. Can this additional information be used in order to decide whether the problems with the integration (with respect to τ) of the quasiperiodic function of τ in the right hand side of (11.75), which are mentioned in Subsection 5.2, really occur? \oslash

11.6 Chaplygin

The last part of Chaplygin [9, §5], starting with “Now we discuss the curves traced out on the surface of the sphere ...”, corresponds to our Remark 11.1.

The proof that the elliptic coordinates (11.21) for u lead to a motion on the Jacobi variety of the hyperelliptic curve (11.50) with constant velocity is contained in Chaplygin [9, §3 up to (30)]. The notations in Chaplygin [9, §3] correspond to ours according to the following list, which is a continuation of the list in Subsection 3.5.

Chaplygin [9, §3 up to (30)]	our notation
(14)	(3.7)
(15)	(3.9), (11.29) and (3.11)
(16) and (17)	(11.30)
(18)	(11.31) and (11.30)
(19)	formulas following (11.40)
$a^2, b^2, c^2, \partial^2, g, k, \sigma$ in (20)	a_1, a_2, a_3 (cf. (11.19)), $1/\rho, 8T, 4\ j\ ^2, 1$
x, y, z in (21)	x_1, x_2, x_3 in (11.19)
(22)	(11.36), (11.31), (11.32), (11.34), (11.35)
(23)	formulas preceding (11.42)
(24)	(11.25) and (11.27)
(25)	(11.26)
(26)	(11.28)
formulas between (26) and (27)	(11.38), (11.40), (11.42)
(27), $j, -g \partial^2/j$	(11.43), $8T - 4\ j\ ^2/\rho, \gamma$
(28)	(11.45)
(29)	(11.53)
(30)	(11.47)
(33)	(11.70)
(34)	(11.71)

In our Subsection 11.2 we have followed the miraculous calculations of Chaplygin [9, §3 up to (30)] quite closely, adding some more explanations in the hope to make these easier to read. In (11.45) we identified (up to a constant factor) the factor

$$\sqrt{(\lambda_2 + 1/\rho) (\lambda_3 + 1/\rho)}$$

introduced in Chaplygin [9, (28)] with the integrating factor $X(u)^{1/2}$ of Lemma 7.1, the same as the factor \sqrt{X} at the end of [9, §2], and of Corollary 8.4. In Subsection 11.4 we added a discussion of the relation between the phase space of the rotational motion and the Jacobi variety of the hyperelliptic curve, about which Chaplygin did not say anything in [9].

Chaplygin did not tell how he came to the idea of using the elliptic coordinates (11.21). The last part of Chaplygin [9, §5] indicates that he had calculated the branch locus of the projection $(u, v) \mapsto u$ from the (j, T) -level surface onto the u -sphere. Therefore he might have observed that in the complex domain it is equal to the union of two quadrics which, together with the sphere, belong to a one-parameter family of confocal quadrics, and this might have prompted him to use the elliptic coordinates (11.21). He might have refrained from mentioning this in his article, because in his time the use of elliptic coordinates in the presence of families of confocal quadrics was standard.

Although Chaplygin described the motion of $u(t)$ in the annulus on the sphere in [9, end of §5], he did not observe that $u(t)$ reaches the boundary curves of the annulus precisely when $d\langle p(t), j \rangle / dt = 0$, cf. Remark 11.11.

12 A Geometric Interpretation

The kinetic energy equation in the form $Y^2 - XZ = 0$, cf. (3.14), is equal to the discriminant equation for the quadratic equation $X(u)\lambda^2 + 2Y(u, v)\lambda + Z(v) = 0$ in the variable λ . Let α be an auxiliary parameter. Using (3.9), the equation for λ can be written in the form

$$\langle \lambda u - v, (J + \alpha)(\lambda u - v) \rangle = (\rho^{-1} + \alpha)\lambda^2 - 2\alpha j_3 \lambda + 2T + \alpha \|j\|^2. \quad (12.1)$$

The discriminant of the right hand side is equal to zero if and only if

$$(\|j\|^2 - j_3^2) \alpha^2 + (2T + \|j\|^2/\rho) \alpha + 2T/\rho = 0. \quad (12.2)$$

If (12.2) holds, then the right hand side of (12.1) is equal to

$$(\rho^{-1} + \alpha) \left\{ \lambda - \alpha j_3 / (\rho^{-1} + \alpha) \right\}^2,$$

and the equation (12.1) is equivalent to

$$\langle v, [\rho(I + \rho)^{-1} + \alpha] v \rangle = \rho^{-1} + \alpha, \quad (12.3)$$

with $v = \theta u - \eta v$ and

$$\theta = \lambda / \left\{ \lambda - \alpha j_3 / (\rho^{-1} + \alpha) \right\}, \quad \eta = 1 / \left\{ \lambda - \alpha j_3 / (\rho^{-1} + \alpha) \right\}.$$

It follows that $\eta = (\theta - 1)(\rho^{-1} + \alpha)/\alpha j_3$, and the conclusion is that the straight line \tilde{l} passing through u , with the direction vector equal to $\alpha j_3 u - (\rho^{-1} + \alpha) v$, is tangent to the quadric Q defined by the equation (12.3). Note that $\tilde{l} = A^{-1}(l)$, where l is equal to the straight line passing through $-e_3$, with direction vector equal to $\alpha j_3 e_3 - (\rho^{-1} + \alpha) j$. Here the rotation $A \in \text{SO}(3)$ varies in the level surface in $\text{SO}(3)$, corresponding to the equations (3.9) and (3.14). Also note that $-Q = Q$, and therefore the same properties hold with l replaced by $-l$.

The equation (12.2) has two solutions α_1 and α_2 , leading to two pairs of straight lines $\pm l_1$ and $\pm l_2$ and quadrics Q_1 and Q_2 to which $A^{-1}(\pm l_1)$ and $A^{-1}(\pm l_2)$ are tangent, respectively. The inner product of the two direction vectors is equal to

$$\begin{aligned} & \langle \alpha_1 j_3 u - (\rho^{-1} + \alpha_1) v, \alpha_2 j_3 u - (\rho^{-1} + \alpha_2) v \rangle \\ &= (\alpha_1/\rho + \alpha_2/\rho + \alpha_1 \alpha_2) (\|j\|^2 - j_3^2) + \|j\|^2/\rho^2. \end{aligned}$$

On the other hand it follows from (12.2) that

$$(\|j\|^2 - j_3^2) (\alpha_1 + \alpha_2) = -2T - \|j\|^2/\rho, \quad (\|j\|^2 - j_3^2) \alpha_1 \alpha_2 = 2T/\rho,$$

and we conclude that the direction vectors are perpendicular. Equivalently, the four straight lines $\pm l_1, \pm l_2$ form a rectangle.

The above arguments work under the assumption that the moment j is neither vertical nor horizontal. If $j \neq 0$ is horizontal, $j_3 = 0$, then the right hand side of (12.1) is a square if $\alpha = -2T\rho/\|j\|^2$, and a constant if $\alpha = -1/\rho$. In the first case the line passing through u with direction vector v is tangent to the quadric defined by (12.3), with $\alpha = -2T\rho/\|j\|^2$. In the second case the line passing through $-v$ with the direction vector u is tangent to the quadric $\langle v, [(I + \rho)^{-1} - \rho^{-1}] v \rangle = 2T - \|j\|^2/\rho$.

Question 12.1 What are *all* the straight lines l in the plane spanned by e_3 and j , and quadrics Q , such that for each $A \in \text{SO}(3)$ in the level surface corresponding to the equations (3.9) and (3.14) we have that $A^{-1}(l)$ is tangent to Q ? This question may be related to Question 11.10. \oslash

12.1 Chaplygin

Section 12 corresponds to Chaplygin's [9, §5]. Chaplygin multiplied the figures by the radius r of the sphere, in order to have the corner point $-re_3$ of the rectangle $\pm l_1, \pm l_2$ attached to the point of contact of the sphere with the plane.

References

- [1] R. Abraham and J.E. Marsden: *Foundations of Mechanics*. Benjamin/Cummings, London, etc., 1978.
- [2] M. Adler and P. van Moerbeke: The algebraic integrability of the geodesic flow on $\text{SO}(4)$. *Invent. math.* **67** (1982) 296–326.
- [3] V.I. Arnol'd and A. Avez: *Ergodic Problems of Classical Mechanics*. W.A. Benjamin, Inc., New York, Amsterdam, 1968.
- [4] V.I. Arnol'd (ed.): *Dynamical Systems III*. Encyclopedia of Mathematical Sciences, vol. 3. Springer-Verlag, New York, 1987.
- [5] D.K. Bobylev: On a sphere with a gyroscope inside. *Mathematical Collection of the Moscow Mathematical Society* **16** (1892) 544–581. (In Russian. A review in German appeared in *Fortschritte der Mathematik* **24** (1892), p. 892.)
- [6] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden, R.M. Murray: Nonholonomic mechanical systems with symmetry. *Arch. Rat. Mech. Anal.* **136** (1996) 21–99
- [7] R. Bott: Nondegenerate critical manifolds. *Annals of Math.* **60** (1954) 248–261.

- [8] S.A. Chaplygin: On a generalization of the theorem of areas with application to the problem of rolling spheres. *Mathematical Collection of the Moscow Mathematical Society* **20** (1897) 1–32. Also pp. 39–71 in *Analysis of the Dynamics of Nonholonomic Systems*. Series on Classical Natural Sciences, Moscow, 1949, and pp. 434–454 in *Selected Works on Mechanics and Mathematics*. State Publishing House, Technical-Theoretical Literature, Moscow, 1954. (All in Russian. A review in German appeared in *Fortschritte der Mathematik* **27** (1896), p. 625, 626.)
- [9] S.A. Chaplygin: On a sphere rolling on a horizontal plane. *Mathematical Collection of the Moscow Mathematical Society* **24** (1903) 139–168. Also pp. 72–99 in *Analysis of the Dynamics of Nonholonomic Systems*. Series on Classical Natural Sciences, Moscow, 1949, and pp. 455–471 in *Selected Works on Mechanics and Mathematics*. State Publishing House, Technical-Theoretical Literature, Moscow, 1954. (All in Russian. A review in German appeared in *Fortschritte der Mathematik* **34** (1903), p. 782. An English translation appeared in *Regular and Chaotic Dynamics* **7** No. 2 (2002) 131–148.)
- [10] Wei-Liang Chow: On compact complex analytic varieties. *Amer. J. Math.* **71** (1949) 893–914.
- [11] R.H. Cushman: Reduction, Brouwer’s Hamiltonian, and the critical inclination. *Celestial mechanics* **31** (1983) 401–429, correction: **33** (1984) p. 395.
- [12] R.H. Cushman and L.M. Bates: *Global Aspects of Classical Integrable Systems*. Birkhäuser Verlag, Basel, Boston, Berlin, 1997.
- [13] H.M. Farkas and I. Kra: *Riemann Surfaces*. Springer-Verlag, New York, Heidelberg, Berlin, 1980.
- [14] G. van der Geer and B. Moonen: *Abelian Varieties*. In preparation.
<http://turing.wins.uva.nl/~b.moonen/boek/BookAV.html>
- [15] Ph. Griffiths and J. Harris: *Principles of Algebraic Geometry*. John Wiley and Sons, 1978.
- [16] V. Guillemin and A. Pollack: *Differential Topology*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, USA, 1974.
- [17] R. Hartshorne: *Algebraic geometry*. GTM, vol. 52, Springer-Verlag, New York, 1977.
- [18] J. Hermans: A symmetric sphere rolling on a surface. *Nonlinearity* **8** (1995) 493–515.
- [19] W.V.D. Hodge and D. Pedoe: *Methods of Algebraic Geometry, vol. I and II*. Cambridge University Press, 1968.
- [20] L. Hörmander: Fourier integral operators I. *Acta Math.* **127** (1971) 79–183.

- [21] C.G.J. Jacobi: *Vorlesungen über Dynamik* (1842-43). Mathematische Werke Vol. VIII, 2nd edition, Chelsea Publ. Co., New York, 1969. (The first edition appeared in 1866.)
- [22] A.A. Kilin: The dynamics of Chaplygin ball: the qualitative and computer analysis. *Regular and Chaotic Dynamics* **6**, No. 3 (2001) 291–306.
- [23] A.N. Kolmogorov: On dynamical systems with integral invariant on the torus. *Dokl. Akad. Nauk SSSR* **93**, No. 5 (1953) 763–766. (Russian). Zbl. **52**, p. 319. M.R. **16**, p. 36.
- [24] P.D. Lax: Integrals of non-linear equations of evolution and solitary waves. *Comm. Pure Appl. Math.* **21** (1968) 467–490.
- [25] S. Lie (unter Mitwirkung von F. Engel): *Theorie der Transformationsgruppen, zweiter Abschnitt*. B.G. Teubner Verlag, Leipzig u. Berlin, 1890, 1930.
- [26] S. Łojasiewicz: *Introduction to Complex Analytic Geometry*. Birkhäuser Verlag, Basel, Boston, Berlin, 1991.
- [27] T. Matsusaka: On a characterization of a Jacobian variety. *Memoirs of the College of Science, University of Kyoto, Series A* **32** (1959) 1–19, Correction **33** (1960/61) 350.
- [28] P. van Moerbeke: The spectrum of Jacobi matrices. *Invent. math.* **37** (1976) 45–81.
- [29] J. Moser: On the volume element on a manifold. *Trans. A.M.S.* **120** (1965) 286–294.
- [30] D. Mumford: *Abelian Varieties*. Oxford University Press, New York, 1974.
- [31] D. Mumford: *Tata Lectures on Theta I*. Birkhäuser, Boston, Basel Stuttgart, 1983.
- [32] D. Mumford: *Tata Lectures on Theta II*. Birkhäuser, Boston, Basel Stuttgart, 1984.
- [33] E. Noether: Invariante Variationsprobleme. *Nachr. v. d. Ges. d. Wiss. zu Göttingen* (1918), 235–257 = *Gesammelte Abhandlungen*, pp. 248–270
- [34] E.J. Routh: *Advanced Dynamics of a System of Rigid Bodies* 6th Edition. MacMillan Company, London, 1905. Reprinted by Dover Publications, New York, 1955.
- [35] I.R. Shafarevich: *Basic Algebraic Geometry*. Springer-Verlag, Berlin, Göttingen, heidelberg, 1977.
- [36] C.L. Siegel: Note on differential equations on the torus. *Annals of Math.* **46** (1945) 423–428
- [37] N.E. Zhukovsky: On Bobylev’s gyroscopic sphere. *Physics Section of the Imperial Friends of Physics, Anthropology and Ethnography, Moscow* **6** (1893) 11–18. (In Russian. A review in German appeared in *Fortschritte der Mathematik* **25** (1893–94), p. 1441.)